SYSTEMS OF VARIATIONAL INEQUALITIES AND MULTIOBJECTIVE GAMES

THESIS

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Dedicated
to my
Parents

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Certificate

This is to certify that the thesis entitled "Systems of variational inequalities and multiobjective games" submitted for the award of the Degree of Doctor of Philosophy in Mathematics of Aligarh Muslim University, Aligarh embodies the original research work carried out by Mr. Zubair Khan under my guidance and supervision and has not been submitted for the award of any other degree or diploma of this or any other university.

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Preface

In 1950s, J. Nash published three papers [105, 106, 107] on several fundamental important aspects of the game theory. He pointed out the distinction between cooperative and non-cooperative games and introduced the concept of equilibrium point, which is now usually known as Nash equilibria. By using Kakutani fixed point, he established several existence results for Nash equilibria of finite cooperative and non-cooperative games. In 1952, Debreu [54] further studied the existence of Nash equilibria of an n-person game with compact strategy sets in $\mathbb{R}^n$ which extended earlier work of Nash [106]. This work of Debreu generalizes the existence theorems on equilibria of the general economic model presented by von Neumann [129] in 1937 in the game theory. Since than, the existence of equilibrium point of cooperative and non-cooperative games, now known as Nash equilibrium problem and Debreu type equilibrium problem, have been studied by many researchers either by using Ky Fan inequality or some fixed point theorems, see for example [138] and references therein. It is mentioned by J. P. Aubin in his book [23] that the Nash equilibrium problem [105] for differentiable functions can be formulated in the form of a variational inequality problem defined over the product of sets. Further, Pang [112] showed that not only Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems from operations research, economics, game theory, mathematical physics and other areas, can also be uniformly modelled as a variational inequality problem defined over the product of sets which is equivalent to the problem of system of variational inequalities. In 1999, Ansari and Yao [18] used a fixed point theorem for a family of multivalued maps to prove the existence of a solution of system of variational inequalities. Since than system of scalar (vector) variational inequalities and system of generalized scalar (vector) variational inequalities are used as tool to prove the existence of a solution of Nash equilibrium problem for differentiable and non-differentiable functions, respectively.

This thesis deals with the existence theory of system of variational inequalities, system of generalized varaiational inequalities, system of generalized scalar (vector) quasi-variational inequalities and their applications to Nash equilibrium problem and Debreu type equilibrium problem for vector valued functions, that is, constrained multiobjective games.

Chapter 1 deals with the brief introduction of scalar (vector) variational inequalities, scalar (vector) variational-like inequalities, scalar (vector) quasi-variational inequalities, scalar (vector) quasi-variational-like inequalities and their generalizations besides some basic definitions and results from nonlinear analysis.

In Chapter 2, we consider the system of variational inequalities which is equivalent to the variational inequalities over the product of sets. By introducing the concept of different kinds of pseudomonotonicities of operators, we establish the existence of a solution of our problems. As an application of our results, we prove the existence of a coincidence point of two families of nonlinear operators.

In Chapter 3, we consider the systems of generalized variational inequalities and generalized variational inequality problems over the product of sets. It is noticed that the problem of system of generalized variational inequalities is equivalent to the generalized variational inequality problems over the product of sets. Various monotonicities defined in Chapter 2 have been extended for multivalued maps. By adopting the technique of Yang and Yao [135] and using the results of Chapter 2, we prove several existence results for a solution of generalized variational inequality problems over the product of sets.

In Chapter 4, we introduce weighted variational inequalities over the product of sets and system of weighted variational inequalities. We notice that the weighted variational inequality problem over the product of sets and the problem of system of weighted variational inequalities are equivalent. We give a relationship between system of weighted variational inequalities and systems of vector variational inequalities. We define several kinds of weighted monotonicities and establish several existence results for a solution of above mentioned problems under these weighted monotonicities. We also introduced the weighted generalized variational inequalities over the product of sets, that is, weighted variational inequalities for multivalued maps. The extensions of weighted monotonicites for multivalued maps are also considered. By adopting the technique used in Chapter 3 to prove the existence of a solution of weighted generalized variational inequalities over the product of sets.

In Chapter 5, we define a weighted Nash equilibrium and a normalized weighted Nash equilibrium of a constrained multiobjective game. A relationship between a weighted Nash equilibrium of a constrained multiobjective game and an optimal solution of a constrained optimization problem is also given. We study the existence of weighted Nash equilibria and Pareto equilibria for the constrained multiobjective games with or without involving Φ-condensing maps. Our results improve and unify the corresponding results of the multiobjective games given in the literature.

In the last chapter, we introduce the concept of system of generalized vector quasi-equilibrium problems which includes system of generalized vector equilibrium problems, system of generalized implicit vector quasi-variational inequalities and system of vector quasi-equilibrium problems as special cases. By using known maximal element theorems for a family of multivalued maps, we prove the existence of a solution of system of generalized vector quasi-equilibrium problems. Several special cases are also discussed. As applications of our results, we derive the existence results for a solution of Debreu type equilibrium problem for vector-valued functions, that is, constrained multiobjective games.

Some portion of Chapters 2 and 3 has already been published in *Journal of Inequalities in Pure and Applied Mathematics* Vol. 4, Issue 1, Article 6, (2003) and Chapters 5 and 6 have been accepted for the publication in *Southeast Asian Bulletin* *of Mathematics* and *Proceedings of the International Conference on Analysis and Discrete structures Editor. S. Nanda, Narosa Publication, New Delhi*, respectively.

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Zubair Khan

Chapter 1

Preliminaries

In this chapter, we first present some basic definitions and results from nonlinear analysis which will be used in the sequel. Then we give a brief introduction of scalar (vector) variational inequalities, scalar (vector) variational-like inequalities and their generalizations. In the last of this chapter a brief introduction of scalar (vector) equilibrium problems and their generalizations is also given.

1.1 Introduction

For the last four decades, nonlinear analysis has been popularized among the researchers due to the applications in Science, Engineering and Social Sciences. The theory of variational inequalities, which was introduced in sixties by the Italian and French Schools as a joint and concerted efforts of two leading mathematicians of that period, Guido Stampacchia and Jacques-Louis Lions, is one of the area of applications of nonlinear analysis.

The KKM theory [138] and fixed point theory in topological vector spaces are the main tools to study the variational inequalities and their generalizations.

In this chapter we recall some basic definitions from nonlinear analysis and present Fan-KKM lemma, some generalizations of Browder fixed point theorem [36] and a maximal element theorem for a family of multivalued maps. A brief introduction of scalar (vector) variational inequalities, scalar (vector) variational-like inequalities and their generalizations, and scalar (vector) equilibrium problems and their generalizations are given.

1.2 Some Basic Definitions and Results From Nonlinear Analysis

In 1929, Knaster, Kuratowski and Mazurkiewicz [81] formulated the so-called KKM principle in the finite dimensional Euclidean space. Later, in 1961, it has been generalized to infinite dimensional Hausdorff topological vector spaces by Ky Fan [63]. Fan also established an elementary but very basic geometric lemma for multivalued maps which is called Fan's geometric lemma. In 1968, Browder gave a fixed point version of Fan's geometric lemma and this result is known as Fan-Browder fixed point theorem. Since than there have been numerous generalizations of Fan-Browder fixed point theorem and their applications to coincidence and fixed point theory, minimax inequalities, variational inequalities, convex analysis, game theory, mathematical economics, social sciences, and so on.

It is well known that the famous Browder fixed point theorem [36] is equivalent to a maximal element theorem (see [136]). In the last decade, many generalized forms of Browder fixed point theorem are used to establish the maximal element theorems for a family of multivalued maps. Such kind of maximal element theorems are useful to establish the existence of a solution of abstract economies or generalized games, system of variational inequalities, etc; See for example [42, 55, 56, 57, 95, 96, 101, 102, 136, 138] and references therein. The Browder fixed point theorem has also been generalized for a family of multivalued maps with applications to maximal elements theory, generalized games or abstract economies, system of variational inequalities, etc; See for example [7, 18, 55, 57, 138] and references therein.

In this section, we recall some basic definitions from nonlinear analysis and present Fan-KKM lemma and some generalizations of Browder fixed point theorem.

For every nonempty set A, we denote by 2^A (respectively, $\mathscr{F}(A)$) the family of all subsets (respectively, finite subsets) of A. If A is a nonempty subset of a vector space, then coA denotes the convex hull of A.

Let X and Y be any nonempty subsets and $T : X \to 2^Y$ a multivalued map. The *graph of* T, denoted by $\mathcal{G}(T)$, is

$$\mathcal{G}(T) = \{(x, y) \in X \times Y : x \in X,\ y \in T(x)\}.$$

The *inverse* T^{-1} of T is the multivalued maps from $\mathcal{R}(T)$, the range of T, to X defined by

$$x \in T^{-1}(y) \quad \text{if and only if} \quad y \in T(x).$$

Definition 1.2.1. Let X and Y be topological vector spaces. A multivalued map $T : X \to 2^Y \setminus \{\emptyset\}$ is said to be *closed* if its graph is closed in $X \times Y$.

Definition 1.2.2. [30] Let X and Y be topological vector spaces. A multivalued map $T : X \to 2^Y \setminus \{\emptyset\}$ is called *upper semicontinuous on* X if T has compact values and for each $x_0 \in X$ and for any open set V in Y containing $T(x_0)$ there exists an open neighborhood U of x_0 in X such that $T(x) \subseteq V$ for all $x \in U$.

Lemma 1.2.1. *[30] Let $T : X \to 2^Y$ be an upper semicontinuous multivalued map on X and D a compact subset of X. Then $T(D)$ is compact.*

Lemma 1.2.2. *[30] If a multivalued maps $T : X \to 2^Y$ is upper semicontinuous on X then it is closed.*

We present the following particular form of Fan-KKM lemma (see [63]).

Theorem 1.2.3. *Let K be a compact and convex subset of a Hausdorff topological vector space X and $K_0 \subseteq K$ be nonempty. Assume that $T : K^0 \to 2^K \setminus \{\emptyset\}$ be a multivalued satisfying the following conditions:*

(i) For each $x \in K^0$, $T(x)$ is closed;

(ii) For every finite set $\{x^1, \ldots, x^m\}$ of K^0 one has $co\{x^1, \ldots, x^m\} \subseteq \bigcup_{k=1}^m T(x^k)$.

Then $\bigcap_{x \in K^0} T(x) \neq \emptyset$.

Definition 1.2.3. [116, 117] Let E be a Hausdorff topological vector space and L a lattice with least element, denoted by $\mathsf{0}$. A mapping $\Phi : 2^E \to L$ is called a *measure of noncompactness* provided that the following conditions hold for any $M, N \in 2^E$:

(i) $\Phi(M) = \mathsf{0}$ if and only if M is precompact (i.e., it is relatively compact).

(ii) $\Phi(\overline{\text{co}}M) = \Phi(M)$, where $\overline{\text{co}}M$ denotes the closed convex hull of M.

(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

It follows from (iii) that if $M \subseteq N$, then $\Phi(M) \leq \Phi(N)$.

Definition 1.2.4. [116, 117] Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A multivalued map $T : D \to 2^E$ is called Φ-*condensing* provided that if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$ then M is relatively compact.

Remark 1.2.1. Note that every multivalued map defined on a compact set is necessarily Φ-condensing. If E is locally convex, then a compact multivalued map (i.e., $T(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : D \to 2^E$ is Φ-condensing and if $T' : D \to 2^E$ satisfies $T'(x) \subseteq T(x)$ for all $x \in D$, then T' is also Φ-condensing.

We mention the following generalization of Browder fixed point theorem due to Chowdhury and Tan [48].

Theorem 1.2.4. *Let K be a nonempty and convex subset of a topological vector space (not necessarily Hausdorff) X and $T : K \to 2^K$ a multivalued map. Assume that the following conditions hold:*

(i) For all $x \in K$, $T(x)$ is convex.

(ii) For each $A \in \mathscr{F}(K)$ and for all $y \in coA$, $T^{-1}(y) \cap coA$ is open in coA.

(iii) For each $A \in \mathscr{F}(K)$ and all $x, y \in coA$ and every net $\{x_\alpha\}_{\alpha\in\Gamma}$ in K converging to x such that $ty + (1-t)x \notin T(x_\alpha)$ for all $\alpha \in \Gamma$ and for all $t \in [0,1]$, we have $y \notin T(x)$.

(iv) There exist a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$ for all $x \in K \setminus D$.

(v) For all $x \in D$, $T(x)$ is nonempty.

Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

Definition 1.2.5. A subset B of a topological space E is said to be *compactly open* (respectively, *compactly closed*) in E if for any nonempty compact subset D of E, $B \cap D$ is open (respectively, closed) in D.

Now we present another generalized form of Browder fixed point theorem due to Chowdhury and Tan [49].

Theorem 1.2.5. *Let K be a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E and $S, T : K \to 2^K$ multivalued maps. Assume that the following conditions hold:*

(i) For all $x \in K$, $S(x) \subseteq T(x)$.

(ii) For all $x \in K$, $T(x)$ is convex and $S(x)$ is nonempty.

(iii) For all $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is comapctly open.

(iv) There exist a nonempty, closed and compact (not necessarily, convex) subset D of K and a $\tilde{y} \in D$ such that $K \setminus D \subset T^{-1}(\tilde{y})$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.

Remark 1.2.2. If K is a nonempty closed convex subset of a Hausdorff topological vector space X, then the condition (iii) of Theorem 1.2.5 can be replaced by the following condition (see, for example, Corollary 2 in [95]).

(iii)$'$ The multivalued map $S : K \to 2^K$ is Φ-condensing.

Definition 1.2.6. Let X and Y be topological vector spaces. A point $\bar{x} \in X$ is said to be a *maximal element* of a multivalued maps $T : X \to 2^Y$ if $T(\bar{x}) = \emptyset$.

In the last of this section we present the following maximal element results for a family of multivalued maps.

Theorem 1.2.6. *[56] Let I be any index set. For each $i \in I$, let K_i be a nonempty and convex subset of a Hausdorff topological vector space X_i, and let $S_i : K = \prod_{i\in I} K_i \to 2^{K_i}$ be a multivalued map. Assume that the following conditions hold:*

(i) For each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex.

(ii) For each $i \in I$ and for all $x \in K$, $x_i \notin S_i(x)$, where x_i is the ith component of x.

(iii) For each $i \in I$ and for all $y_i \in K_i$, $S_i^{-1}(y_i)$ is open in K.

(iv) There exist a nonempty and compact subset N of K and a nonempty, compact and convex subset B_i of K_i for each $i \in I$, such that for all $x \in K \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$.

Then there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Remark 1.2.3. If for each $i \in I$, K_i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, then condition (iv) of Theorem 1.2.6 can be replaced by the following condition:

(iv)′ The multivalued map $S : K \to 2^K$ defined as $S(x) := \prod_{i \in I} S_i(x)$ for all $x \in K$, is Φ-condensing.

(See, Corollary 4 in [42]).

1.3 Variational Inequalities and their Generalizations

The theory of variational inequalities is a powerful tool of the current mathematical technology, introduced by G. Stampacchia and G. Fichera separately, in early sixties. The ideas and techniques of variational inequalities are being applied in various fields of mathematics, engineering and social sciences including fluid flow through porous media, contact problems in elasticity, optimal control, nonlinear optimization, transportation and economics equilibrium, etc.; See for example [25, 47, 67, 72, 80, 85, 111, 115] and references therein.

In the last two decades various extensions and generalizations of variational inequalities has been proposed and analyzed. Quasi-variational inequalities are the extended form of variational inequalities in which the underlying convex set does depend upon the solution. These were introduced and studied by Bensoussan, Goursat, and Lions [28]. For further details we refer to Baiocchi and Capelo [25], Bensoussan [27], Bensoussan and Lions [29] and Mosco [103].

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and K be a nonempty, convex subset of E. Given a mapping $f : E \to E^*$, then the *classical variational* *inequality problem* is the following:

$$\text{(VIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that} \\ \langle f(x), y - x \rangle \geq 0, \text{ for all } y \in K. \end{cases} \tag{1.3.1}$$

Inequality (1.3.1) is called *variational inequality.* Every solution of (VIP) is a solution of an optimization problem for a convex and differentiable function ϕ and vice-versa with $f(x) = \phi'(x)$. If the function ϕ is not convex then for studying optimization problem, we need the following generalization of (VIP).

Let $\eta(.,.) : K \times K \to E$ be a bifunction. The *variational-like inequality problem* is the following:

$$\text{(VLIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that} \\ \langle f(x), \eta(y, x) \rangle \geq 0, \text{ for all } y \in K. \end{cases} \tag{1.3.2}$$

Inequality (1.3.2) is called *variational-like inequality.* It is first studied in 1989 by Parida, Sahoo and Kumar [113] in the setting of n-dimensional Euclidean space $\mathbb{R}^n$ in the study of mathematical programming problem. It has been further studied by Yang and Chen [134] in the study of economic equilibrium problems, and Siddiqi, Khaliq and Ansari [125] in the setting of reflexive Banach spaces and topological vector spaces. Recently, Ansari and Yao [22] have studied such type of problems by using the auxiliary variational principle technique and suggested an iterative algorithms.

In many applications, the convex set in the formulation of (VIP) also depends upon the solution itself. In this case (VIP) is called *quasi-variational inequality problem.* More precisely, for a given multivalued map $Q : K \to 2^K$, the *quasi-variational inequality problem* is the following:

$$\text{(QVIP)} \quad \begin{cases} \text{Find } x \in Q(x) \text{ such that} \\ \langle f(x), y - x \rangle \geq 0, \text{ for all } y \in Q(x). \end{cases} \tag{1.3.3}$$

We also define *quasi-variational-like inequality problem* as follows:

$$\text{(QVLIP)} \quad \begin{cases} \text{Find } x \in Q(x) \text{ such that} \\ \langle f(x), \eta(y, x) \rangle \geq 0, \text{ for all } y \in Q(x). \end{cases} \tag{1.3.4}$$

To study the optimization problem for nondifferentiable but convex functions, we use the generalized variational inequality problem, that is, variational inequality problem for multivalued map. Fang and Peterson [64] first introduced the following problem.

For a given multivalued map $T : E \to 2^{E^*}$, the *generalized variational inequality problem* is defined as follows:

$$\text{(GVIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, y - x \rangle \geq 0, \text{ for all } y \in K. \end{cases} \tag{1.3.5}$$

If the underlying function ϕ in the formulation of an optimization problem is nondifferentiable and nonconvex, then one can not use the (GVIP). We need to define the following *generalized variational-like inequality problem*:

$$\text{(GVLIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, \eta(y, x) \rangle \geq 0, \text{ for all } y \in K. \end{cases} \tag{1.3.6}$$

We remark that every solution of (GVLIP) and nondifferentiable and nonconvex optimization problems are equivalent; See for example [71] and references therein.

For further generalizations of variational-like inequalities, we refer to [13].

1.4 Vector Variational Inequalities

The classical variational inequality for vector valued functions is known as a *vector variational inequality*; it has been introduced by F. Giannessi [70] in 1980 with further applications in finite dimensional spaces. Since then, the theory of vector variational inequalities has emerged as a new direction for researchers. Because of the applications in Sciences, Engineering and Social Sciences, it has been studied and generalized by many authors; see for example [71] and references therein. In the recent past, vector variational inequalities and their generalizations are used as tools to study the vector optimization problems for nondifferentiable (in some sense) and nonconvex functions, see for example [21] and references therein.

Let Y be a topological vector space. A nonempty subset P of Y is called a *convex cone* if $P + P = P$ and $\alpha P \subseteq P$ for all $\alpha > 0$. P is called a *pointed cone* if P is a cone and $P \cap (-P) = \{0\}$. The *Partial order* in Y is denoted by $\leq$ and is defined as $x \leq y$ if and only if $y - x \in P$ for all $x, y \in Y$, and in this case P is called a *positive cone* in Y. Further more, if such a partial order is induced by a convex cone, it is called a *linear order.* An ordered topological vector space is a pair (Y, P) where Y is a topological vector space and P is a pointed convex cone with linear order induced by P. The *weak order* $\not<$ on ordered topological vector space (Y, P) with int $P \neq \phi$ is defined as $x \not< y$ if and only if $y - x \notin \text{int } P$ for all $x, y \in Y$, where int P denotes the interior of P.

Let X and Y be two topological vector spaces and K be a nonempty subset of X. Let $f : K \to L(X, Y)$ be a map, where $L(X, Y)$ is the space of all continuous linear operators from X into Y. Let $C : K \to 2^Y$ be a multivalued map such that for each $x \in K$, $C(x)$ is a closed pointed convex cone in Y with int $C(x) \neq \phi$ for all $x \in K$. Then the *vector variational inequality problem* is the following:

$$\text{(VVIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle f(x_0), y - x_0 \rangle \notin -\text{int } C(x_0), \quad \text{for all } y \in K. \end{cases} \tag{1.4.1}$$

Where $\langle f(x), y \rangle$ denotes the evaluation of the linear operator $f(x)$ at y and hence $\langle f(x), y \rangle \in Y$. (1.4.1) is called a *vector variational inequality.* It has been studied by many researchers in the setting of infinite dimensional spaces with or without monotonicity assumptions, see for example a recent monograph [71] edited by F. Giannessi and references therein.

In many applications, the convex set in the formulation of (VVIP) also depends upon the solution itself. In this case (VVIP) is called *vector quasi-variational inequality problem.* More precisely, for a given multivalued map $Q : K \to 2^K$, the *vector quasi-variational inequality problem* is the following:

$$\text{(VQVIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle f(x_0), y - x_0 \rangle \notin -\text{int } C(x_0), \quad \text{for all } y \in Q(x). \end{cases}$$

Let $\eta : K \times K \to X$ be a continuous map. Then Siddiqi et al [124] have studied the following *vector variational-like inequality problem*:

$$\text{(VVLIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle f(x_0), \eta(y, x_0) \rangle \notin -\text{int } C(x_0), \quad \text{for all } y \in K. \end{cases} \tag{1.4.2}$$

When $C(x)$ is a fixed cone, that is, $C(x) = P$ then for all $x \in K$ and (Y, P) is an ordered topological vector space with weak order, (VVIP) becomes the following problem:

$$\text{(VVIP)}' \quad \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle f(x_0), y - x_0 \rangle \not< 0, \quad \text{for all } y \in K. \end{cases} \tag{1.4.1$'$}$$

This problem has been investigated by Chen and Cheng [43], Chen and Yang [45] and Yang [132]. They have also studied the relations among the (VVIP), vector complementarity problem, fixed point problem and vector optimization problem.

In this case, the (VVLIP) reduces to the following problem:

$$\text{(VVLIP)}' \quad \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle f(x_0), \eta(y, x_0) \rangle \not< 0, \quad \text{for all } y \in K. \end{cases} \tag{1.4.2$'$}$$

Yang [133] has considered (VVLIP) and established its equivalence with a generalized multiobjective programming problem.

Let $F : K \to 2^{L(X,Y)}$ be a multivalued map and $C(x)$ is a fixed cone, that is, $C(x) = P$ for all $x \in K$, then the (VVIP) is called the *generalized vector variational inequality problem* which is defined as follows:

$$\text{(GVVIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ and } s_0 \in F(x_0) \text{ such that} \\ \langle s_0, y - x_0 \rangle \notin -\text{int } P, \quad \text{for all } y \in K. \end{cases} \tag{1.4.3}$$

It has been studied by Chen and Craven [44] and Lee et al [89, 92] with applications in optimization.

Let $F : K \to 2^{L(X,Y)}$ be a multivalued map, then (VVLIP) is known as the *generalized vector variational-like inequality problem* studied by Ansari [1, 2, 3] and Lee et al [90]:

$$\text{(GVVLIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ and } s_0 \in F(x_0) \text{ such that} \\ \langle s_0, \eta(y, x_0) \rangle \notin -\text{int } C(x_0), \quad \text{for all } y \in K. \end{cases} \tag{1.4.4}$$

If we take $Y = \mathbb{R}$, $C(x) := \mathbb{R}_+$, for all $x \in K$, then the (VVI), (VVLI), (GVVI) and (GVVLI) reduce to the variational inequality problem [80], variational-like inequality problem [16, 113, 125, 134], generalized variational inequality problem [64, 120, 122] and the generalized variational-like inequality problem [123], respectively. For $C(x)$ is a fixed cone, the weak formulation of (GVVLIP) is considered and studied by Ansari and Siddiqi [12]. They obtained relationships among vector optimization problem, weak formulation of (GVVLIP) and optimization problem of a utility function over a set of efficient points, by introducing the concept of η-subgradient and η-subdifferential.

More precisely, for a given multivalued map $Q : K \to 2^K$, the *generalized vector quasi-variational inequality problem* is the following:

$$\text{(GVQVIP)} \quad \begin{cases} \text{Find } x_0 \in K \text{ and } s_0 \in F(x_0) \text{ such that} \\ \langle s_0, y - x_0 \rangle \notin -\operatorname{int} C(x_0), \quad \text{for all } y \in Q(x). \end{cases}$$

1.5 Equilibrium Problems

Let K be a nonempty convex subset of a topological vector space X and let $f : K \times K \to \mathbb{R}$ be a bifunction such that $f(x, x) \geq 0$ for all $x \in K$. The *equilibrium problem* (in short, EP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \text{for all } y \in K,$$

which includes optimization, saddle point, fixed point, complementarity and variational inequality problems as special cases. The (EP) has been studied by many authors, see for example [15, 33, 34, 37, 39, 40, 41, 74, 77, 82, 96, 138] because of its general form and it includes many fundamental mathematical problems from optimization, operations research, economics, mechanics and engineering. Since (EP) is a unified model of numerous important problems from nonlinear analysis, optimization, game theory, complementarity systems, engineering, physics, social sciences, etc., many of the results obtained for (EP) provides the same results in nature for the problems stated above.

Let $A : K \to 2^K$ be a multivalued map with nonempty values. The *quasi-equilibrium problem* (in short, QEP) considered and studied by Ding [59] and Lin and Park [97] is to find $\bar{x} \in K$ such that

$$\bar{x} \in A(\bar{x}) \text{ and } f(\bar{x}, y) \geq 0, \text{ for all } y \in A(\bar{x}).$$

Inspired by the extension of variational inequality problems for vector valued functions by Giannessi [70], in the recent past (EP) has been extended for vector valued bifunction in [4, 9, 32, 71, 74, 91, 96, 108, 128] with applications in vector optimization problem, vector saddle point problem and Nash equilibrium problem for vector valued functions.

Let X and Y be real topological vector spaces and K a nonempty subset of X. Let C be an ordered cone in Y, that is, a closed and convex cone in Y with int $C \neq \phi$, where int C denotes the topological interior of C. For a given vector valued bifunction $f : K \times K \to Y$, the *vector equilibrium problem* (for short, VEP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\text{int } C, \quad \text{for all } y \in K,$$

which is a unified model of several known problems, for instance, vector variational inequality and variational-like inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems; See, for example, [4, 26, 32, 68, 71, 74, 91, 108, 128] and references therein. For a more comprehensive bibliography on vector equilibrium problems, vector variational inequality and variational-like inequality problems and their generalizations, we refer to a recent volume [71] edited by F. Giannessi.

For a more generalized form of (VEP) which includes vector quasi-variational inequality problem (for short, VQVIP), vector quasi-optimization problem (for short, VQOP) and vector quasi-saddle point problem (for short, VQSPP) as special cases, we let $A : K \to 2^K$ be a multivalued map with nonempty values, then we consider the following problem:

$$\text{(VQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \bar{x} \in A(\bar{x}) : \ f(\bar{x}, y) \notin -\text{int } C, \quad \text{for all } y \in A(\bar{x}). \end{cases}$$

It is known as *vector quasi-equilibrium problem* (for short, VQEP) and introduced in [17]. Some existence results for a solution to (VQEP) and consequently for (VQVIP), (VQOP) and (VQSPP) have been established in [17].

In [93] (respectively, [17]) (VVIP) (respectively, VQVIP) is used as a tool to solve (VOP) (respectively, VQOP) for differentiable (in some sense) vector valued functions. The (VOP) for nondifferentiable vector valued functions can be solved by using generalized vector variational inequality problems. For further details, we refer to [21] and references therein. To obtain a more general problem which contains (VEP) and generalized vector variational inequality problems as special cases, we consider the function f to be multivalued, that is, $F : K \times K \to 2^Y \setminus \{\phi\}$ and in this case, (VEP) can be generalized in the following way:

$$\text{(GVEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ F(\bar{x}, y) \nsubseteq -\text{int } C, \quad \text{for all } y \in K. \end{cases}$$

It is called *generalized vector equilibrium problem* (for short, GVEP) and it has been studied by many authors; See, for example, [8, 14, 19, 88] and references therein.

For the *generalized vector equilibrium problem* [8, 9, 19, 88, 109, 110, 126], which includes (VEP) and generalized vector variational inequality problems as special cases, one replaces the range space $\mathbb{R}$ by a real topological vector space Y with an ordering cone C that is, C is a proper, closed, convex and solid cone in Y, and one considers a multivalued map $F : K \times K \to 2^Y \setminus \{\phi\}$. Then the inequality $f(x, y) \geq 0$ occurring in (EP) may be generalized in several possible ways, for instance as $F(x, y) \subseteq -C$, $F(x, y) \cap C \neq \phi$, $F(x, y) \cap -\text{int } C = \phi$, $F(x, y) \nsubseteq C$ and $F(x, y) \nsubseteq -\text{int } C$.

In the very first paper on generalized vector equilibrium problems, Ansari, Oettli and Schlager [9] considered the following forms:

$$\text{(GVEP1)} \quad \begin{cases} \text{Find } \bar{x} \in M \text{ such that} \\ F(\bar{x}, y) \cap P, \quad \text{for all } y \in N, \end{cases}$$

and

$$\text{(GVEP2)} \quad \begin{cases} \text{Find } \bar{x} \in M \text{ such that} \\ F(\bar{x}, y) \subseteq P, \quad \text{for all } y \in N, \end{cases}$$

where M and N are nonempty subsets of topological vector spaces X and Z, respectively, P is a nonempty subset of a topological vector space Y and $F : M \times N \to 2^Y \setminus \{\phi\}$ is a multivalued map.

With $P = Y \setminus \{\text{-int } C\}$, $X = Z$ and $M = N = K$, the (GVEP1) and (GVEP2) are also studied by Song [126] and in this case (GVEP1) contains the following problem:

$$\text{(GVEP3)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ F(\bar{x}, y) \nsubseteq -\text{int } C, \quad \text{for all } y \in K, \end{cases}$$

With $P = C$, $X = Z$ and $M = N = K$, the (GVEP2) reduces to the following problem:

$$\text{(GVEP4)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ F(\bar{x}, y) \subseteq C, \quad \text{for all } y \in K, \end{cases}$$

Oettli and Schlager [110] considered the following form of generalized vector equilibrium problem with moving ordering cone:

$$\text{(GVEP5)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ F(\bar{x}, y) \nsubseteq C(\bar{x}), \quad \text{for all } y \in K, \end{cases}$$

where $C : K \to 2^Y$ are multivalued maps such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with int $C(x) \neq \phi$, where int $C(x)$ denotes the interior of $C(x)$.

We mention the following form of *generalized vector equilibrium problem* which is a unified and general model of most of the vector variational inequalities and their generalized forms studied in the literature:

$$\text{(GVEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ F(\bar{x}, y) \nsubseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in K, \end{cases}$$

where K is a nonempty convex subset of a topological vector space X, Y is another topological vector space, $F : K \times K \to 2^Y \setminus \{\phi\}$ and $C : K \to 2^Y$ are multivalued maps such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with int $C(x) \neq \phi$.

This problem is considered by Ansari et al [8, 19], Konnov and Yao [88], and Oettli and Schlager [109].

We also mention the following *generalized vector quasi-equilibrium problem* (for short, GVQEP) which is a unified format of generalized vector quasi-variational inequality problems, generalized vector quasi-variational-like inequality problems, etc.

$$\begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \bar{x} \in A(\bar{x}) : F(\bar{x}, y) \nsubseteq -\text{int } C, \quad \text{for all } y \in A(\bar{x}), \end{cases}$$

For a more general form of (GVQEP), we replace the ordered cone C by a "moving cone". More precisely, we consider a multivalued map $C : K \to 2^Y$ such that for each $x \in K$, $C(x)$ is a proper, closed and convex cone with int $C(x) \neq \phi$, then the (GVQEP) can be written in the following form:

$$\text{(GVQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \bar{x} \in A(\bar{x}) : \; F(\bar{x}, y) \nsubseteq -\text{int } C(\bar{x}), \quad \text{for all } y \in A(\bar{x}). \end{cases}$$

Chapter 2

Systems of Variational Inequalities

In this chapter, we consider the system of variational inequalities which is equivalent to the variational inequalities over the product of sets. We introduce the concept of relatively quasi monotone and densely relative pseudomonotone operators. We also define the relatively B-pseudomonotonicity and relatively demimonotonicity which generalize the well-known pseudomonotonicity in the sense of Brézis [36]. Several existence results for a solution of our problems are established. As an application of our results, we prove the existence of a coincidence point of two families of nonlinear operators.

2.1 Introduction

In the recent past, systems of variational inequalities are used as tools to solve various equilibrium-type problems, like, Nash equilibrium, traffic equilibrium, spatial equilibrium and general equilibrium programming problems, from operations research, economics, game theory, mathematical physics and other areas, see for example [18, 23, 51, 65, 83, 84, 100, 104, 112] and references therein. Pang [112] uniformly modeled these equilibrium-type problems in the form of a variational inequality defined on a product of sets . He decomposed the original variational inequality into a system of variational inequalities, which are easy to solve, to establish some solution methods for finding the approximate solutions of above mentioned equilibrium-type problems. He also studied the convergence of such solutions. The decomposition method is also used by many other authors, see for example [51, 65, 83, 84, 100] and references therein. Later, it is found that these two problems, variational inequality defined on a product of sets and system of variational inequalities, are equivalent.

Recently, Konnov [86] noticed that the solution sets of variational inequalities over product of sets are invariant with respect to certain affine transformations of cost mappings. Taking these as a basis, he introduced new concept of (pseudo) monotonicity, called *relatively (pseudo) monotonicity*, which are adjusted for a decomposable structure of the initial problem. By using the famous Fan-KKM lemma [63], he proved some existence results for a solution of variational inequalities over product of sets under these relatively monotonicities.

In the next section, we formulate problem of system of variational inequalities and variational inequality problem over product of sets. We notice that these two problems are equivalent. Inspired by the work of Luc [99], in section third, we introduce the concept of relatively quasimonotonicity and densely relatively pseudomonotonicity which are much weaker than the relatively pseudomonotonicity considered by Konnov [86]. We also define the relatively B-pseudomonotonicity and relatively demimonotonicity which extend in a natural way the well-known pseudomonotonicity in the sense of Brézis [36] (see also [37]). In Section 4, we establish some existence results for a solution of variational inequality over product of sets and hence for a solution of system of variational inequalities under these monotonicities. Last section is devoted to the applications of our results. By using the results of Section 4, we derive the existence of a coincidence point of two families of nonlinear operators.

2.2 Formulations

In this section, we give the formulations of a system of variational inequalities and variational inequality problem over product of sets. We also notice that these problems are equivalent.

Let I be a finite index set, that is, $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let X_i be a Hausdorff topological vector space with its dual X_i^*, K_i a nonempty convex subset of X_i, $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$, and $X^* = \prod_{i\in I} X_i^*$. For each $i \in I$, when X_i is a normed space, its norm is denoted by $||\cdot||_i$ and the product norm on X will be denoted by $||\cdot||$. We denote by $\langle\cdot,\cdot\rangle$ the pairing between X_i^* and X_i. For each $x \in X$, we write $x = (x_i)_{i\in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the ith component of x. For each $i \in I$, let $f_i : K \to X_i^*$ be a nonlinear operator. We consider the following *problem of system of variational inequalities*, which is the model of various equilibrium-type problems from operations research, economics, game theory, mathematical physics and other areas, see for example, [18, 65, 104, 112] and references therein:

$$\text{(SVI)} \quad \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } \ i \in I, \\ \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } \ y_i \in K_i, \end{cases}$$

It is easy to see that (SVI) is equivalent to the following *variational inequality problem over product of sets* (for short, VIPPS): Find $\bar{x} \in K$ such that

$$\sum_{i\in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } \ y_i \in K_i, \ \ i \in I. \tag{2.2.1}$$

It is mentioned by J. P. Aubin in his book [23] that the Nash equilibrium problem [105] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets

Of course, if we define the mapping $f : K \to X^*$ by

$$f(x) = \big(f_i(x)\big)_{i\in I}, \tag{2.2.2}$$

then (VIPPS) can be equivalently re-written as the usual variational inequality problem of finding $\bar{x} \in K$ such that

$$\langle f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } \ y \in K. \tag{2.2.3}$$

Konnov [86] introduced the concept of relatively pseudomonotonicity and strongly relatively pseudomonotonicity to prove some existence results for a solution of (VIPPS) in the setting of Banach spaces. Konnov [87] also studied combined relaxation method for solving (VIPPS). He essentially exploited the decomposable structure of (2.2.3) and simplified their implementation. He also noted that the method can not be extended directly due to its two-level structure and a binding condition in its line search procedure.

2.3 Relatively Monotonicities

In this section, we first recall the definition of relatively (strictly) pseudomonotonicity, introduced by Konnov [86], and then we introduce the concept of relatively (strictly) quasimonotonicity and relatively densely (strictly) monotonicity. In the last of this section we also define the relatively B-pseudomonotonicity and relatively demimonotonicity which extend in a natural way the well-known pseudomonotonicity in the sense of Brézis [36].

Definition 2.3.1. The map $f : K \to X^*$, defined by (2.2.2), is said to be

(i) *relatively pseudomonotone at* $y \in K$ [86] if for all $x \in K$, we have

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0 \quad \Rightarrow \quad \sum_{i\in I}\langle f_i(y), y_i - x_i\rangle \geq 0,$$

and *relatively strictly pseudomonotone at* $y \in K$ if the second inequality is strict for all $x \neq y$;

(ii) *relatively quasimonotone at* $y \in K$ if for all $x \in K$, we have

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle > 0 \quad \Rightarrow \quad \sum_{i\in I}\langle f_i(y), y_i - x_i\rangle \geq 0.$$

If f is relatively pseudomonotone (respectively, relatively strictly pseudomonotone and relatively quasimonotone) at each $y \in K$, then we say that it is relatively pseudomonotone (respectively, relatively strictly pseudomonotone and relatively quasimonotone) on K.

Of course, if I is a singleton set, then Definition 2.3.1 (ii) reduces to the usual definition of quasimonotonicity, see for example, [73, 75, 78, 79].

Definition 2.3.2. The map $f : K \to X^*$, defined by (2.2.2), is said to be *hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle f(x + \lambda(y - x)), y - x\rangle$ is continuous.

Lemma 2.3.1. *Let f, defined by (2.2.2), be hemicontinuous and relative quasimonotone on K. Then for every $x, y \in K$ with $\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0$ we have either*

$$\sum_{i\in I}\langle f_i(y), y_i - x_i\rangle \geq 0 \quad \text{or} \quad \sum_{i\in I}\langle f_i(x), z_i - x_i\rangle \leq 0 \quad \text{for all} \quad z_i \in K_i \quad i \in I.$$

Proof. It is sufficient to show that if for all $z_i \in K_i$, $i \in I$,

$$\sum_{i \in I} \langle f_i(x), z_i - x_i \rangle > 0,$$

then we have

$$\sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \geq 0.$$

Let us set $y^t = tz + (1 - t)y$ for $0 < t \leq 1$. Then, obviously, $y^t \in K$ and

$$\sum_{i \in I} \langle f_i(x), y_i^t - x_i \rangle > 0.$$

By relatively quasimonotonicity of f, we get

$$\sum_{i \in I} \langle f_i(y^t), y_i^t - x_i \rangle \geq 0.$$

Now let $t \to 0$. Since $y^t \to y$ along a line segment, and by hemicontinuity of f, we have

$$\sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \geq 0.$$

This completes the proof. □

Remark 2.3.1. If the index set I is singleton, then Lemma 2.3.1 reduces to Lemma 3.1 (ii) in [73].

Definition 2.3.3. [99] A subset K^0 of K is said to be *segment-dense in* K if for all $x \in K$, there can be found $x^0 \in K^0$ such that x is a cluster point of the set $[x, x^0] \cap K^0$, where $[x, x^0]$ denotes the line segment joining x and x^0 including end points.

Definition 2.3.4. [99] For each $i \in I$, let K_i be a nonempty convex subset of X_i. For each $i \in I$, we set

$$K_i^{\perp} := \{\xi_i \in X_i^* : \langle \xi_i, y_i - x_i \rangle = 0 \quad \text{for all} \quad x_i, y_i \in K_i\}$$

and call it the *orthogonal complement of* K_i. Then

$$\begin{aligned} K^{\perp} &:= \prod_{i \in I} K_i^{\perp} = \prod_{i \in I} \{\xi_i \in X_i^* \ : \ \langle \xi_i, y_i - x_i \rangle = 0 \ \text{ for all } x_i, y_i \in K_i\} \\ &= \{\xi := (\xi_i)_{i \in I} \in X^* : \ \text{for each } i \in I, \ \langle \xi_i, y_i - x_i \rangle = 0 \ \text{ for all } x_i, y_i \in K_i\} \end{aligned}$$

Remark 2.3.2. For a given $\xi_i \in X_i^*$, the following two statements are equivalent:

(a) For each $i \in I$, $\langle \xi_i, y_i - x_i \rangle = 0$ for all $x_i, y_i \in K_i$;

(b) $\sum_{i\in I} \langle \xi_i, y_i - x_i \rangle = 0$ for all $x_i, y_i \in K_i, \quad i \in I$.

Indeed, (a) implies (b) is obvious. For (b) implies (a), let $y_j = x_j$ for $j \neq i$, in (b) then we obtain (a).

In view of above remark, we have

$$K^{\perp} = \{\xi = (\xi_i)_{i\in I} \in X^* : \sum_{i\in I} \langle \xi_i, y_i - x_i \rangle = 0 \quad \text{for all} \quad x_i, y_i \in K_i, \quad i \in I\}$$

and we call it the *orthogonal complement of* K.

Definition 2.3.5. Let f be a map from K to X^* defined by (2.2.2). We say that $x^0 \in K$ is a *positive point of* f *on* K if for all $x \in K$ one has either $f(x) \in K^{\perp}$, that is, for each $i \in I$, $f_i(x) \in K_i^{\perp}$ or there exists $y \in K$ such that

$$\sum_{i\in I} \langle f_i(x), y_i - x_i^0 \rangle > 0.$$

The set of all positive points of f on K is denoted by K_f.

We denote by $f(K)$ the image of K under f, that is, $f(K) = \{f(x) : x \in K\}$.

Proposition 2.3.2. *Let f, defined by (2.2.2), be hemicontinuous and relatively quasimonotone on K such that $f(K) \cap K^{\perp} = \emptyset$, that is, for each $i \in I$, $f_i(K) \cap K_i^{\perp} = \emptyset$. Then f is relatively pseudomonotone at every positive point.*

Proof. Let $y \in K_f$ and $x \in K$ be any point such that $\sum_{i\in I} \langle f_i(x), y_i - x_i \rangle \geq 0$. Then by Lemma 2.3.1, we have either

$$\sum_{i\in I} \langle f_i(y), y_i - x_i \rangle \geq 0 \quad \text{or} \quad \sum_{i\in I} \langle f_i(x), z_i - x_i \rangle \leq 0 \quad \text{for all } z_i \in K_i, \quad i \in I. \tag{2.3.1}$$

To complete the proof, it is sufficient to show that the second inequality in (2.3.1) is impossible.

Indeed, since $y \in K_f$ and for each $i \in I$, $f_i(x) \notin K_i^{\perp}$, then there exists $z \in K$ such that

$$\sum_{i \in I} \langle f_i(x), z_i - y_i \rangle > 0.$$

Then

$$\sum_{i \in I} \langle f_i(x), z_i - x_i \rangle = \sum_{i \in I} \langle f_i(x), z_i - y_i \rangle + \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle > 0$$

which shows that the second inequality in (2.3.1) is impossible, and the proof is completed. $\square$

Proposition 2.3.3. *Let K be closed and convex subset of X and K^0 a segment-dense subset of K. If f, defined by (2.2.2), is relatively quasimonotone at every point of K^0 and hemicontinuous on K, then it is relatively quasimonotone on K.*

Proof. Let $x, y \in K$ with

$$\sum_{i \in I} \langle f_i(x), y_i - x_i \rangle > 0. \tag{2.3.2}$$

Since K^0 is a segment-dense subset of K, we can find $y^0 \in K^0$ and $y^m \in [y, y^0] \cap K^0$, for all $m \in \mathbb{N}$ such that $\lim y^m = y$. Then from (2.3.2), we obtain

$$\sum_{i \in I} \langle f_i(x), y_i^m - x_i \rangle > 0, \quad \text{for all } \ m \in \mathbb{N}.$$

Since f is relatively quasimonotone at y^m, we get

$$\sum_{i \in I} \langle f_i(y^m), y_i^m - x_i \rangle \geq 0.$$

Since $\lim y^m = y$ and by hemicontinuity of f, we have

$$\sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \geq 0.$$

Hence f is relatively quasimonotone on K. $\square$

Now we are ready to define a new concept of densely relative pseudomonotonicity, which generalize the notion of densely pseudomonotonicity considered by Luc [99].

Definition 2.3.6. The map $f : K \to X^*$, defined by (2.2.2), is said to be *densely relatively pseudomonotone (respectively, densely relatively strictly pseudomonotone) on* K if there exists a segment-dense subset $K^0 \subseteq K$ such that f is relatively pseudomonotone (respectively, relatively strictly pseudomonotone) on K^0.

Next, we define the relatively B-pseudomonotonicity and relatively demimonotonicity which extend the definition of a pseudomonotone map, introduced by Brézis [36].

Definition 2.3.7. The map $f : K \to X^*$, defined by (2.2.2), is said to be *relatively B-pseudomonotone* (respectively, *relatively demimonotone*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha\in\Gamma}$ in K converging to x (respectively, weakly to x) with

$$\liminf_\alpha\Big[\sum_{i\in I}\langle f_i(x^\alpha), x_i - x_i^\alpha\rangle\Big] \geq 0$$

we have

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq \limsup_\alpha\Big[\sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle\Big] \quad \text{for all } \; y \in K.$$

Of course, if I is a singleton set, then the above definition reduces to the definition of a pseudomonotone map, introduced by Brézis [36] (see also [37] and [[23], p. 410]).

2.4 Existence Results

This section deals with the existence results for a solution of (VIPPS) and hence for (SVI) under the assumption of various monotonicities defined in the previous section.

The following lemma can be treated as a generalization of Minty lemma (see for example, [[80] Chapter 3, Lemma 1.5]) to (VIPPS).

Lemma 2.4.1. *Let K be a nonempty convex subset of X and K^0 be the same as in the definition of densely relatively pseudomonotone map. If f, defined by (2.2.2), is hemicontinuous and densely relatively pseudomonotone, then the following problem is equivalent to (VIPPS):*

$$(MVIPPS)^0 \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that} \\ \sum_{i\in I}\langle f_i(y), y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } \; y_i \in K_i^0, \;\; i \in I. \end{cases}$$

The solution sets of (VIPPS) and $(\text{MVIPPS})^0$ are denoted by K_s and K_{sm}^0, respectively.

Proof. By the densely relatively pseudomonotonicity of f, we have $K_s \subseteq K_{sm}^0$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{MVIPPS})^0$. Then

$$\sum_{i \in I} \langle f_i(y), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } \; y_i \in K_i^0, \;\; i \in I. \tag{2.4.1}$$

Since K^0 is segment-dense, for all $z \in K$, we can find $z^0 \in K^0$ and $z^m \in [z, z^0] \cap K^0$ for all $m \in \mathbb{N}$ such that $\lim z^m = z$. Then from (2.4.1), we get

$$\sum_{i \in I} \langle f_i(z^m), z_i^m - \bar{x}_i \rangle \geq 0, \quad \text{for all } \; m \in \mathbb{N}.$$

Since $\lim z^m = z$ and f is hemicontinuous, we obtain

$$\sum_{i \in I} \langle f_i(z), z_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } \; z_i \in K_i, \;\; i \in I.$$

Again by hemicontinuity of f (see the proof of Lemma 2 in [86]), we have

$$\sum_{i \in I} \langle f_i(\bar{x}), z_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } \; z_i \in K_i, \;\; i \in I.$$

Hence $\bar{x} \in K_s$ and thus $K_s = K_{sm}^0$. □

Theorem 2.4.2. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and f, defined by (2.2.2), be hemicontinuous and densely relatively pseudomonotone on K. Then (VIPPS) has a solution and hence (SVI) has a solution.*

Proof. Let K^0 be the same as in the definition of a densely relatively pseudomonotone map. For each $y \in K^0$, define two multivalued maps $S, T : K^0 \to 2^K$ by

$$S(y) = \{x \in K : \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle \geq 0\}$$

and

$$T(y) = \{x \in K : \sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \geq 0\}.$$

Then for each $y \in K^0$, $T(y)$ is closed, and also by relatively pseudomonotonicity of f on K^0, we have $S(y) \subseteq T(y)$. By using the standard argument, it is easy to see that for every finite set $\{y^1, \ldots, y^m\}$ of K^0 one has $\text{co}\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m S(y^k)$ (see for example, the proof of Theorem 1 in [86]). Since for all $y \in K^0$, $S(y) \subseteq T(y)$, we also have, $\text{co}\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m T(y^k)$. By applying Theorem 1.2.3, we have $\bigcap_{y \in K^0} T(y) \neq \emptyset$, that is, there exists $\bar{x} \in K$ such that

$$\sum_{i \in I} \langle f_i(y), y_i - x_i \rangle \geq 0, \quad \text{for all } \ y_i \in K_i^0, \ \ i \in I.$$

By Lemma 2.3.1, $\bar{x} \in K$ is a solution of (VIPPS). □

Corollary 2.4.3. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and f, defined by (2.2.2), be hemicontinuous and relatively quasimonotone on K such that K_f is segment-dense in K. Then (VIPPS) has a solution and hence (SVI) has a solution.*

Proof. Let $\bar{x} \in K$ such that $f(\bar{x}) \in K^\perp$, then $\sum_{i \in I} \langle f_i(x), y_i - \bar{x}_i \rangle = 0$ for all $y_i \in K_i$, $i \in I$. Hence $\bar{x} \in K$ is a solution of (VIPPS). Therefore, we may assume that $f(K) \cap K^\perp = \emptyset$. Then by Proposition 2.3.2, f is relatively pseudomonotone at every point of K_f. Since K_f is segment-dense in K, f is densely relatively pseudomonotone on K. Thus by Theorem 2.4.2, (VIPPS) has a solution. □

Corollary 2.4.4. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and f, defined by (2.2.2), be hemicontinuous and densely relatively strictly pseudomonotone on K. Then (VIPPS) has a solution $\bar{x} \in K$, and it is unique if $\bar{x} \in K^0$. Further, $\bar{x} \in K$ is a solution of (SVI), and it is unique if $\bar{x} \in K^0$, where K^0 is the same as in the definition of a densely relatively pseudomonotone map.*

Proof. In view of Theorem 2.4.2, it is sufficient to show that (VIPPS) has at most one solution. Assume to the contrary that $x', x'' \in K^0$ are two solutions of (VIPPS) such that $x' \neq x''$. Then

$$\sum_{i \in I} \langle f_i(x'), x_i'' - x' \rangle \geq 0,$$

By densely relatively strictly pseudomonotonicity of f on K^0, we have

$$\sum_{i\in I}\langle f_i(x''), x_i'' - x'\rangle > 0, \quad \text{i.e.} \quad \sum_{i\in I}\langle f_i(x''), x_i' - x''\rangle < 0.$$

Thus x'' is not a solution of (VIPPS), which is a contradiction of our assumption. This completes the proof. □

Corollary 2.4.5. *For each $i \in I$, let X_i be a real reflexive Banach space and K_i a nonempty, closed and convex subset of X_i. Let f, defined by (2.2.2), be hemicontinuous and densely relatively pseudomonotone on K. Then under each of the following conditions (VIPPS) has a solution.*

For each sequence $\{x^m\} \subseteq K$ with $||x^m|| \to \infty$ as $m \to \infty$,

(h1) there exists $m_0 > 0$ such that $\sum_{i\in I}\langle f_i(x^{m_0}), x_i^{m_0}\rangle \geq 0$;

(h2) there exist $m_0 > 0$ and $y \in K$ with $||y|| < ||x^{m_0}||$ such that $\sum_{i\in I}\langle f_i(x^{m_0}), y_i - x_i^{m_0}\rangle \leq 0$;

(h3) there exist $m_0 > 0$ and $y \in K$ such that $\sum_{i\in I}\langle f_i(y), x_i^m - y_i\rangle > 0$, for all $m \geq m_0$.

Proof. For each $i \in I$, we denote by $B_i(m) = \{x_i \in K_i : ||x_i||_i \leq m\}$ the closed ball with center at 0 and radius $m \in \mathbb{N}$ in K_i, and $B(m) = \prod_{i\in I} B_i(m)$ for all $m \in \mathbb{N}$. Then for each $i \in I$ and for all $m \in \mathbb{N}$, $B_i(m)$ is nonempty, compact and convex. By Theorem 2.4.2, there exists $x_i^m \in K_i$ for each $i \in I$ such that

$$\sum_{i\in I}\langle f_i(x^m), y_i - x_i^m\rangle \geq 0, \quad \text{for all } y_i \in B_i(m), \; i \in I \text{ and for every } m \in \mathbb{N}. \quad (2.4.2)$$

Set

$$g(y) = \sum_{i\in I}\langle f_i(x^m), y_i\rangle, \quad \text{for all } y \in K.$$

Then, clearly g is linear and hence convex. If $||x^m|| < m$ for some m, then by (2.4.2), we have

$$g(x^m) \leq g(y), \quad \text{for all } y \in B(m).$$

Thus x^m is a local minimum of a convex function g, hence it is a global minimum, that is,

$$g(x^m) \leq g(y), \quad \text{for all } \ y \in K,$$

that is,

$$\sum_{i \in I} \langle f_i(x^m), y_i - x_i^m \rangle \geq 0, \quad \text{for all } \ y \in K,$$

which means that x^m is a solution of (VIPPS).

If $||x^m|| = m$ for all $m \in \mathbb{N}$. Assume that condition (h1) holds. Then we show that x^{m_0} is a solution of (VIPPS).

Indeed, for all $y_i \in K_i$, there is a $t \in (0,1]$ for each $i \in I$ such that $ty_i \in B_i(m_0)$. From (2.4.2), we get

$$\begin{aligned} 0 &\leq \sum_{i \in I} \langle f_i(x^{m_0}), ty_i - x_i^{m_0} \rangle \\ &\leq t \sum_{i \in I} \langle f_i(x^{m_0}), y_i - x_i^{m_0} \rangle - (1-t) \sum_{i \in I} \langle f_i(x^{m_0}), x_i^{m_0} \rangle \end{aligned}$$

By condition (h1), we obtain

$$\sum_{i \in I} \langle f_i(x^{m_0}), y_i - x_i^{m_0} \rangle \geq 0, \quad \text{for all } \ y_i \in K_i, \ \ i \in I.$$

Hence x^{m_0} is a solution of (VIPPS).

Under condition (h2) and by using (2.4.2), we obtain

$$\sum_{i \in I} \langle f_i(x^{m_0}), y_i \rangle = \sum_{i \in I} \langle f_i(x^{m_0}), x_i^{m_0} \rangle,$$

that is,

$$g(y) = g(x^{m_0}), \quad \text{for all } \ x^{m_0} \ \text{ with } \ m_0 = ||x^{m_0}|| > ||y||.$$

It follows that y is a local minimum of g on $B(m_0)$. Consequently, it is a global minimum of g and we obtain $g(y) \leq g(z)$ for all $z \in K$, that is,

$$\sum_{i \in I} \langle f_i(x^{m_0}), z_i \rangle \geq \sum_{i \in I} \langle f_i(x^{m_0}), y_i \rangle = \sum_{i \in I} \langle f_i(x^{m_0}), x_i^{m_0} \rangle.$$

This implies that

$$\sum_{i\in I}\langle f_i(x^{m_0}), z_i - x_i^{m_0}\rangle \geq 0, \quad \text{for all } \ z_i \in K_i, \ \ i \in I.$$

Hence x^{m_0} is a solution of (VIPPS).

Finally, condition (h3) and the relatively quasimonotonicity of f (in view of Proposition 2.3.3) imply

$$\sum_{i\in I}\langle f_i(x^m), y_i - x_i^m\rangle \leq 0, \quad \text{for all } \ m \geq m_0.$$

For m sufficiently large, we have $||y|| < ||x^m||$, that is, condition (h2) holds. Hence (VIPPS) has a solution. □

Now we establish some results on the existence of a solution of (VIPPS) or (SVI) under relatively B-pseudomonotonicity assumption by using a fixed point theorem (Theorem 1.2.4) of Chowdhury and Tan [48].

Theorem 2.4.6. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Let f, defined by (2.2.2), be relatively B-pseudomonotone such that for each $A \in \mathscr{F}(K)$, $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is upper semicontinuous on coA. Assume that there exist a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0$. Then (VIPPS) has a solution and hence (SVI) has a solution.*

Proof. For each $x \in K$, define a multivalued map $T : K \to 2^K$ by

$$T(x) = \{y \in K : \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle < 0\}.$$

Then for all $x \in K$, $T(x)$ is convex. Let $A \in \mathscr{F}(K)$, then for all $y \in coA$,

$$[T^{-1}(y)]^c \cap coA = \{x \in coA : \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0\}$$

is closed in coA by upper semicontinuity of the map $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ on coA. Hence $T^{-1}(y) \cap coA$ is open in coA.

Suppose that $x, y \in coA$ and $\{x^\alpha\}_{\alpha\in\Gamma}$ is a net in K converging to x such that

$$\sum_{i\in I}\langle f_i(x^\alpha), (ty_i + (1-t)x_i) - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma \text{ and all } t \in [0,1].$$

For $t = 0$, we have

$$\sum_{i\in I}\langle f_i(x^\alpha), x_i - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma,$$

and therefore

$$\liminf_\alpha\Big[\sum_{i\in I}\langle f_i(x^\alpha), x_i - x_i^\alpha\rangle\Big] \geq 0.$$

By the relatively B-pseudomonotonicity of f, we have

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq \limsup_\alpha\Big[\sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle\Big]. \tag{2.4.3}$$

For $t = 1$, we have

$$\sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma,$$

and therefore,

$$\liminf_{\alpha\in\Gamma}\Big[\sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle\Big] \geq 0. \tag{2.4.4}$$

From (2.4.3) and (2.4.4), we obtain

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0,$$

and thus $y \notin T(x)$.

Assume that for all $x \in D$, $T(x)$ is nonempty. Then all the conditions of Theorem 1.2.4 are satisfied. Hence there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, that is,

$$0 = \sum_{i\in I}\langle f_i(\hat{x}), \hat{x}_i - \hat{x}_i\rangle < 0,$$

a contradiction. Thus there exists $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$, that is,

$$\sum_{i\in I}\langle f_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } y_i \in K_i \;\; i \in I.$$

Hence $\bar{x}$ is a solution of (VIPPS). $\square$

Corollary 2.4.7. *For each $i \in I$, let K_i be a nonempty, closed and convex subset of a real reflexive Banach space X_i. Let f, defined by (2.2.2), be relatively demimonotone such that for each $A \in \mathscr{F}(K)$, $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is upper semicontinuous on coA. Assume that there exists $\tilde{y} \in K$ such that*

$$\lim_{\|x\|\to\infty,\ x\in K} \sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0. \tag{2.4.5}$$

Then (VIPPS) has a solution and hence (SVI) has a solution.

Proof. Let $\alpha = \lim_{\|x\|\to\infty,\ x\in K}\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0$. Then by (2.4.5), $\alpha < 0$. Let $r > 0$ be such that $\|\tilde{y}\| \le r$ and $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < \frac{\alpha}{2}$ for all $x \in K$ with $\|x\| > r$. For each $i \in I$, let $B_i^r = \{x_i \in K_i : \|x_i\|_i \le r\}$, and we denote by $B^r = \prod_{i\in I} B_i^r$. Then B^r is a nonempty and weakly compact subset of K. Note that for any $x \in K \setminus B^r$, $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < \frac{\alpha}{2} < 0$, and the conclusion follows from Theorem 2.4.6. □

2.5 Some Applications

As applications of results of previous section, in this section, we establish some existence results for a coincidence point of two families of nonlinear operators.

The following result on the existence of a coincidence point of two families of nonlinear operators is derived by using Corollary 2.4.5.

Theorem 2.5.1. *For each $i \in I$, let X_i be a real reflexive Banach space. Let f be defined by (2.2.2) and $g = (g_i)_{i\in I}$ be two nonlinear operators from X to X^* such that $(f - g)$ is hemicontinuous and densely relatively pseudomonotone on X, where for each $i \in I$, $g_i : X \to X_i^*$ is a nonlinear map. Assume that at least one of the following conditions holds:*

For each sequence $\{x^m\} \subseteq X$ with $\|x^m\| \to \infty$ as $m \to \infty$,

(h11) there exists $m_0 > 0$ such that

$$\sum_{i\in I}\langle f_i(x^{m_0}), x_i^{m_0}\rangle \ge \sum_{i\in I}\langle g_i(x^{m_0}), x_i^{m_0}\rangle;$$

(h22) there exist $m_0 > 0$ and $y \in K$ with $||y|| < ||x^{m_0}||$ such that

$$\sum_{i \in I} \langle f_i(x^{m_0}), y_i - x_i^{m_0} \rangle \leq \sum_{i \in I} \langle g_i(x^{m_0}), y_i - x_i^{m_0} \rangle;$$

(h33) there exist $m_0 > 0$ and $y \in K$ such that

$$\sum_{i \in I} \langle f_i(y), x_i^m - y_i \rangle > \sum_{i \in I} \langle g_i(y), x_i^m - y_i \rangle, \quad \text{for all } \; m \geq m_0, \quad i \in I.$$

Then there exists $\bar{x} \in K$ such that $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.

Proof. From the Corollary 2.4.5, there exists $\bar{x} \in X$ such that for each $i \in I$,

$$\langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq \langle g_i(\bar{x}), y_i - \bar{x}_i \rangle, \quad \text{for all } \; y_i \in X_i.$$

Therefore we have, $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$. □

As an application of Corollary 2.4.7, we establish the existence of a coincidence point for two families of nonlinear operators.

Theorem 2.5.2. *For each $i \in I$, let X_i be a real reflexive Banach space. Let $f, g : X \to X_i^*$ be defined as $f(x) = (f_i(x))_{i \in I}$ and $g(x) = (g_i(x))_{i \in I}$, respectively, for all $x \in X$, where for each $i \in I$, $g_i : X \to X_i^*$ is a nonlinear operator. Assume that $(f - g)$ is relatively demimonotone and for each $A \in \mathscr{F}(X)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on coA. Further, assume that there exists $\tilde{y} \in X$ such that*

$$\lim_{||x|| \to \infty, \; x \in X} \sum_{i \in I} \langle (f_i - g_i)(x), \tilde{y}_i - x_i \rangle < 0.$$

Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.

Proof. From the Corollary 2.4.7, there exists $\bar{x} \in X$ such that for each $i \in I$,

$$\langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq \langle g_i(\bar{x}), y_i - \bar{x}_i \rangle, \quad \text{for all } \; y_i \in X_i.$$

Therefore we have, $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$. □

Finally, we give another application of Corollary 2.4.7 in the setting of Hilbert spaces.

Theorem 2.5.3. *For each $i \in I$, let $(X_i, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and K_i a nonempty, closed and convex subset of X_i. Let f, defined by (2.2.2), be relatively demimonotone such that for each $A \in \mathscr{F}(K)$, $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is lower semicontinuous on coA. Assume that there exists $\tilde{y} \in K$ such that*

$$\lim_{||x||\to\infty,\ x\in K} \sum_{i\in I}\langle x_i - f_i(x), \tilde{y}_i - x_i\rangle < 0.$$

Then there exists $\bar{x} \in K$ such that for each $i \in I$, $f_i(\bar{x}) = \bar{x}_i$.

Proof. For each $i \in I$, define a nonlinear operator $S_i : K \to X_i$ by $S_i(x) = x_i - f_i(x)$ for all $x \in K$. Then obviously, for each $i \in I$, S_i satisfies all the conditions of Corollary 2.4.7. Hence there exists $\bar{x} \in K$ such that for each $i \in I$, $\langle S_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0$ for all $y_i \in K_i$. For each $i \in I$, let $y_i = f_i(\bar{x})$, we have $||\bar{x}_i - f_i(\bar{x})|| \leq 0$. Therefore, for each $i \in I$, $f_i(\bar{x}) = \bar{x}_i$. □

Chapter 3

Systems of Generalized Variational Inequalities

In this chapter, we consider the systems of generalized variational inequalities and generalized variational inequality problems over the product of sets. It is noticed that the problem of system of generalized variational inequalities is equivalent to the generalized variational inequality problems over the product of sets. Various monotonicities defined in the previous chapter have been extended for multivalued maps. By adopting the technique of Yang and Yao [135] and using the results of previous chapter, we prove several existence results for a solution of generalized variational inequality problems over the product of sets.

3.1 Introduction

In the last three decades, the Nash equilibrium problem [105] has been studied by many authors either by using Ky Fan minimax inequality [62] or by using fixed point technique. In 2000, Ansari and Yao [20] gave a new direction, called *system of generalized variational inequalities*, to solve the Nash equilibrium problem for non-differentiable functions. They introduced a system of optimization problems which includes the Nash equilibrium problem as a special case. They proved that every solution of system of generalized variational inequalities is also a solution of system of optimization problems for nondifferentiable functions and also for nondifferentiable and nonconvex functions.

Next section deals with the formulations of systems of generalized variational inequalities and generalized variational inequality problem over the product of sets. Since every solution of generalized variational inequality problems over the product of sets is a solution of system of generalized variational inequalities and vice-versa, it is a new approach to study system of generalized variational inequalities via generalized variational inequality problems over the product of sets. In the third section, we extend the various kinds of monotonicities defined in the previous chapter to the multivalued maps. The last section deals with the existence results for a solution of generalized variational inequality problems over the product of sets and hence a solution of system of generalized variational inequalities. We adopt the technique of Yang and Yao [135] to establish the existence of a solution of generalized variational inequality problems over the product of sets or system of variational inequalities by using the existence results for a solution of (VIPPS) or (SVI).

3.2 Formulations

Let I be a finite index set, that is, $I = \{1, 2, \dots, n\}$. For each $i \in I$, let X_i be a topological vector space with its dual X_i^*, K_i a nonempty and convex subset of X_i, $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$, and $X^* = \prod_{i\in I} X_i^*$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between X_i^* and X_i. For each $i \in I$, when X_i is a normed space, its norm is denoted by $|| \cdot ||_i$ and the product norm on X will be denoted by $|| \cdot ||$. For each $x \in X$, we write $x = (x_i)_{i\in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the ith component of x. For each $i \in I$, let $F_i : K \to 2^{X_i^*}$ be a multivalued map with nonempty values so that if we set

$$F = (F_i : i \in I), \tag{3.2.1}$$

then $F : K \to 2^{X^*}$ is a multivalued map with nonempty values. Ansari and Yao [20] introduced and studied the following problem of system of generalized variational inequalities: find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall\, y_i \in K_i, \quad \exists\, \bar{u}_i \in F_i(\bar{x}) \quad : \quad \langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0, \tag{3.2.2}$$

where I is an infinite index set. By using a fixed point for a family of multivalued maps [18], Ansari and Yao [20] proved the existence of a solution of this problem. As applications, they derived several existence results for a solution of Nash equilibrium problem for nondifferentiable functions.

We consider the following more general *problem of system of generalized variational inequalities*:

$$\text{(SGVIP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ and } \bar{u} \in F(\bar{x}) \text{ such that} \\ \langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i, \ i \in I, \end{cases}$$

where u_i is the ith component of u.

It can be easily seen that every solution of (SGVIP) is a solution of (3.2.2), but the converse is not true in general.

Next, we introduce the following *generalized variational inequality problem over the product of sets*:

$$\text{(GVIPPS)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ and } \bar{u} \in F(\bar{x}) \text{ such that} \\ \sum_{i \in I} \langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i, \ i \in I, \end{cases}$$

where u_i is the ith component of u.

As in the single valued case, it is easy to see that (GVIPPS) and (SGVIP) are equivalent, that is, every solution of (GVIPPS) is a solution of (SGVIP) and vice-versa.

3.3 Generalized Relatively Monotonicities

In this section, we define different kinds of relatively monotonicities for the multivalued maps. The definitions in this section are the generalizations of the definitions given in the last chapter.

Definition 3.3.1. The multivalued map $F : K \to 2^{X^*}$, defined by (3.2.1), is said to be

(i) *generalized relatively pseudomonotone at* $y \in K$ if for all $x \in K$ and for all $u \in F(x),\ v \in F(y)$, we have

$$\sum_{i \in I} \langle u_i, y_i - x_i \rangle \geq 0 \quad \Rightarrow \quad \sum_{i \in I} \langle v_i, y_i - x_i \rangle \geq 0,$$

and *generalized relatively strictly pseudomonotone at* $y \in K$ if the second inequality is strict for all $x \neq y$, where u_i and v_i are the ith components of u and v, respectively;

(ii) *generalized relatively quasimonotone at* $y \in K$ if for all $x \in K$ and for all $u \in F(x),\ v \in F(y)$, we have

$$\sum_{i \in I} \langle u_i, y_i - x_i \rangle > 0 \quad \Rightarrow \quad \sum_{i \in I} \langle v_i, y_i - x_i \rangle \geq 0,$$

where u_i and v_i are the ith components of u and v, respectively.;

(iii) *generalized densely relatively pseudomonotone*, (respectively, *generalized densely relatively strictly pseudomonotone*) on K if there exists a segment-dense subset $K^0 \subseteq K$ such that F is generalized relatively pseudomonotone (respectively, generalized relatively strictly pseudomonotone) on K^0.

If F is generalized relatively pseudomonotone (respectively, generalized relatively strictly pseudomonotone and generalized relatively quasimonotone) at each $y \in K$, then we say that it is *generalized relatively pesudomonotone* (respectively, *generalized relatively strictly pseudomonotone* and *generalized relatively quasimonotone*) *on* K.

Of course, if I is a singleton set, then Definition 3.3.1 reduces to the usual definitions of pseudomonotonicity and quasimonotonicity, see for example, [53, 75, 78, 79] and references therein.

If F is a single valued map then Definition 3.3.1 reduces to Definition 2.3.1.

Definition 3.3.2. The multivalued map $F : K \to 2^{X^*}$, defined by (3.2.1), is said to be *u-hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0,1]$ the mapping $\lambda \mapsto \langle F(x+\lambda z), z\rangle$ with $z = y - x$ is upper semicontinuous at 0.

Now we define the generalized relatively B-pseudomonotonicity and generalized relatively demimonotonicity which reduce to the definition of a pseudomonotone map, introduced by Brézis [36].

Definition 3.3.3. The multivalued map $F : K \to 2^{X^*}$, defined by (3.3.1), is said to be *generalized relatively B-pseudomonotone* (respectively, *generalized relatively demimonotone*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha\in\Gamma}$ in K converging to x (respectively, weakly to x) with

$$\liminf_\alpha \Big[\sum_{i\in I}\langle u_i^\alpha, x_i - x_i^\alpha\rangle\Big] \geq 0, \quad \text{for all} \;\; u_i^\alpha \in F_i(x^\alpha)$$

we have, for all $u_i \in F_i(x)$

$$\sum_{i\in I}\langle u_i, y_i - x_i\rangle \geq \limsup_\alpha \Big[\sum_{i\in I}\langle u_i^\alpha, y_i - x_i^\alpha\rangle\Big] \quad \text{for all} \;\; u_i^\alpha \in F_i(x^\alpha) \;\; \text{and} \;\; y \in K.$$

Definition 3.3.4. Let Z be topological vector space with its topological dual Z^* and U a nonempty subset of Z. Let $G : U \to 2^{Z^*}$ be a multivalued map and $g : U \to Z^*$ a single valued map. The map g is called a *selection* of G on U if $g(x) \in G(x)$ for all $x \in U$. Furthermore, the map g is called a *continuous selection* of G on U if it is continuous on U and a selection of G on U.

For further detail on continuous selections of multivalued maps, we refer to [118].

It follows from (2.2.2) and (3.2.1) that if $f : K \to X^*$, defined by (2.2.2), is a selection of $F : K \to 2^{X^*}$, defined by (3.2.1), then for each $i \in I$, $f_i : K \to X_i^*$ is a selection of $F_i : K \to 2^{X_i^*}$ on K.

Lemma 3.3.1. *Let $f : K \to X^*$, defined by (2.2.2), be a selection of a multivalued map $F : K \to 2^{X^*}$, defined by (3.2.1), on K. Then*

(i) if F is generalized relatively pseudomonotone, then f is relatively pseudomonotone;

(ii) if F is generalized densely relatively pseudomonotone, then f is densely relatively pseudomonotone;

(iii) if F is generalized relatively B-pseudomonotone (respectively, generalized relatively demimonotone), then f is relatively B-pseudomonotone (respectively, relatively demimonotone).

3.4 Existence Results for (GVIPPS) and (SGVIP)

In this section, we adopt the technique of Yang and Yao [135] to derive the existence results for a solution of (GVIPPS) and (SGVIP) by using the existence results for a solution of (VIPPS) and (SVI) of Chapter 2.

Lemma 3.4.1. *If $f : K \to X^*$, defined by (2.2.2), is a selection of $F : K \to 2^{X^*}$, defined by (3.2.1), on K and $\bar{x} \in K$ is a solution of (VIPPS), then $(\bar{x}, \bar{u})$ is a solution of (GVIPPS) with $\bar{u}_i = f_i(\bar{x})$ for all $i \in I$.*

Proof. Assume that $\bar{x} \in K$ is a solution of (VIPPS). Then

$$\sum_{i\in I}\langle f_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } y_i \in K_i,\ i \in I.$$

Let $\bar{u}_i = f_i(\bar{x})$, so that $\bar{u} = f(\bar{x})$. Since f is a selection of F, we have $\bar{u} \in F(\bar{x})$ such that

$$\sum_{i\in I}\langle \bar{u}_i, y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } y_i \in K_i,\ i \in I.$$

Hence $(\bar{x}, \bar{u})$ is a solution of (GVIPPS). $\square$

Rest of the section, we shall assume that the pairing $\langle\cdot,\cdot\rangle$ is continuous.

First we prove some existence results for a solution of (GVIPPS) under generalized densely relatively pseudomonotonicity and generalized relatively quasimonotonicity.

Theorem 3.4.2. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of a real Hausdorff topological vector space X_i. Assume that* *(i)* $F : K \to 2^{X^*}$, *defined by (3.2.1), is a generalized densely relatively pseudomonotone multivalued map;*

(ii) there exists a continuous selection $f : K \to X^*$, *defined by (2.2.2), of* F *on* K.

Then (GVIPPS) has a solution.

Proof. It follows from condition (ii) that there exists a continuous function f such that $f(x) \in F(x)$ for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. By Lemma 3.3.1, f is relatively densely pseudomonotone and continuous, and so it is densely relatively pseudomonotone and hemicontinuous. Then all the conditions of Theorem 2.4.2 are satisfied and hence there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then Lemma 3.4.1 implies that $(\bar{x}, \bar{u})$ is a solution of (GVIPPS). □

Theorem 3.4.3. *For each* $i \in I$, *let* K_i *be a nonempty, compact and convex subset of a real Hausdorff topological vector space* X_i. *Assume that*

(i) the multivalued map $F : K \to 2^{X^*}$, *defined by (3.2.1), is generalized relatively quasimonotone on* K;

(ii) there exists a continuous selection $f : K \to X^*$, *defined by (2.2.2), of* F *on* K *such that* K_f *is segment-dense in* K.

Then (GVIPPS) has a solution.

Proof. It is on the lines of the proof of Theorem 3.4.2 by using Corollary 2.4.3. □

Theorem 3.4.4. *For each* $i \in I$, *let* K_i *be a nonempty, closed and convex subset of a real reflexive Banach space* X_i. *Assume that*

(i) the multivalued map $F : K \to 2^{X^*}$, *defined by (3.2.1), is generalized relatively quasimonotone on* K;

(ii) there exists a continuous selection $f : K \to X^*$, *defined by (2.2.2), of* F *on* K *such that* K_f *is segment-dense in* K.

Then under each of the following conditions (GVIPPS) has a solution.

For each sequence $\{x^m\} \subseteq K$ with $||x^m|| \to \infty$ as $m \to \infty$,

(h31) there exists $m_0 > 0$ such that $\sum_{i\in I}\langle u_i^{m_0}, u_i^{m_0}\rangle \geq 0$ for all $u_i^{m_0} \in F_i(x^{m_0})$;

(h32) there exist $m_0 > 0$ and $y \in K$ with $||y|| < ||x^{m_0}||$ such that $\sum_{i\in I}\langle u_i^{m_0}, y_i - x_i^{m_0}\rangle \leq 0$, for all $u_i^{m_0} \in F_i(x^{m_0})$;

(h33) there exist $m_0 > 0$ and $y \in K$ such that $\sum_{i\in I}\langle v_i, x_i^m - y_i\rangle > 0$, for all $m \geq m_0$ and $v_i \in F_i(y)$.

Proof. It is on the lines of the proof of Theorem 3.4.2 by using Corollary 2.4.5. □

Now we prove the existence of a solution of (GVIPPS) under generalized relatively B-pseudomonotonicity.

Theorem 3.4.5. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Assume that*

(i) $F : K \to 2^{X^*}$, defined by (3.2.1), is a generalized relatively B-pseudomonotone multivalued map;

(ii) there exists a continuous selection $f : K \to X^*$, defined by (2.2.2), of F on K;

(iii) there exist a nonempty, closed and compact subset D of K and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0$.

Then (GVIPPS) has a solution.

Proof. From condition (ii), there exists a continuous functions $f : K \to X^*$ such that $f(x) \in F(x)$, for all $x \in K$, that is, for each $i \in I$, $f_i(x) \in F_i(x)$ for all $x \in K$. Since the pairing $\langle\cdot,\cdot\rangle$ is continuous, we have the map $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is continuous on K. By Lemma 3.3.1, f is relatively B-pseudomonotone. Therefore from Theorem 2.4.6, there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then by Lemma 3.4.1, $(\bar{x}, \bar{u})$ is a solution of (GVIPPS). □

Corollary 3.4.6. *For each $i \in I$, let K_i be a nonempty and convex subset of a real reflexive Banach space X_i. Assume that* *(i)* $F : K \to 2^{X^*}$*, defined by (3.2.1), is a generalized relatively demimonotone multivalued map;*

(ii) there exists a continuous selection $f : K \to X^*$*, defined by (2.2.2), of* F *on* K*;*

(iii) there exists $\tilde{y} \in K$ *such that*

$$\lim_{\|x\| \to \infty,\ x \in K} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0.$$

Then (GVIPPS) has a solution.

Proof. It follows from Corollary 2.4.7 by using the argument of Theorem 3.4.5. □

Chapter 4

Weighted Variational Inequalities

In this chapter, we introduce weighted variational inequalities over product of sets and system of weighted variational inequalities. It is noticed that the weighted variational inequality problem over product of sets and the problem of system of weighted variational inequalities are equivalent. We give a relationship between system of weighted variational inequalities and systems of vector variational inequalities. We define several kinds of weighted monotonicities and establish several existence results for a solution of above mentioned problems under these weighted monotonicities. We also introduced the weighted generalized variational inequalities over product of sets, that is, weighted variational inequalities for multivalued maps. The extensions of weighted monotonicites for multivalued maps are also considered. In the last of this chapter, we adopt the technique used in Chapter 3 to prove the existence of a solution of weighted generalized variational inequalities over product of sets.

4.1 Introduction

Decision making is an integral part of our daily lives. It considers situations ranging in complexity from the simple to the most complex involving multiple objectives. An important class of multiobjective decision problems is the vector optimization or multiobjective optimization problems. From a methodological viewpoint, these are mathematical programming problems with a vector valued objective functions. The

Nash equilibrium problem for vector valued functions is a game with multicriteria in its strategy, called multiobjective game. Recently, Ansari, Schaible and Yao [10] studied Nash equilibrium problem for vector valued functions or multiobjective game by using systems of vector variational inequalities and established several existence results for its solution. It was the starting point to study the Nash equilibrium problem for vector valued functions by using systems of vector variational inequalities.

In the next section, we introduce weighted variational inequalities over product of sets and system of weighted variational inequalities and recall the formulations of systems of vector variational inequalities. It is noticed that the weighted variational inequality problem over product of sets and the problem of system of weighted variational inequalities are equivalent. The concept of a normalized solution of system of weighted variational inequalities is introduced and its relationship with the solution of systems of vector variational inequalities are given. In section third, several kinds of weighted monotonicities are defined to establish several existence results for a solution of above mentioned problems. In the last section, we also introduced the weighted generalized variational inequalities over product of sets, that is, weighted variational inequalities for multivalued maps. The extensions of weighted monotonicities for multivalued maps are also considered. We adopt the technique used in Chapter 3 to prove the existence of a solution of weighted generalized variational inequalities over product of sets.

4.2 Formulations and Preliminaries

For each given $m \in \mathbb{N}$, we denote by $\mathbb{R}^m_+$ the non-negative orthant of $\mathbb{R}^m$, that is,

$$\mathbb{R}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_j \geq 0, \quad \text{for } j = 1, \ldots, m\},$$

so that $\mathbb{R}^m_+$ has a nonempty interior with the topology induced in terms of convergence of vectors with respect to the Euclidean metric. That is,

$$\operatorname{int} \mathbb{R}^m_+ = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_j > 0, \quad \text{for } j = 1, \ldots, m\}.$$

We denote by $\mathbb{T}_+^m$ and int $\mathbb{T}_+^m$ the simplex of $\mathbb{R}_+^m$ and its relative interior, respectively, that is,

$$\mathbb{T}_+^m = \{u = (u_1, \ldots, u_m) \in \mathbb{R}_+^m : \sum_{j=1}^{m} u_j = 1\}, \text{ and}$$

$$\text{int } \mathbb{T}_+^m = \{u = (u_1, \ldots, u_m) \in \text{int } \mathbb{R}_+^m : \sum_{j=1}^{m} u_j = 1\}.$$

e denotes the unit vector in $\mathbb{R}^m$, that is $e = (1, \ldots, 1)$.

Let I be a finite index set, that is, $I = \{1, \ldots, n\}$ and for each $i \in I$, let ℓ_i be a positive integer. For each $i \in I$, let X_i be a real topological vector space with its dual X_i^*, K_i a nonempty convex subset of X_i, $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$. For each $x \in X$, $x_i \in X_i$ denotes the ith coordinate and we write $x = (x_i)_{i \in I}$. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $f_{i_j} : K \to X_i^*$ be a map. For each $i \in I$, we denote

$$F_i(x) := \left(f_{i_1}(x), \ldots, f_{i_{\ell_i}}(x)\right) \quad \text{for all} \quad x \in K,$$

and

$$\langle F_i(x), x_i - y_i \rangle := \left(\langle f_{i_1}(x), x_i - y_i \rangle, \ldots, \langle f_{i_{\ell_i}}(x), x_i - y_i \rangle\right),$$

where $\langle \cdot, \cdot \rangle$ denotes the continuous pairing between X_i^* and X_i.

We consider the following *systems of vector variational inequalities*:

$$\text{(SVVI)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } \ i \in I, \\ \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \notin \mathbb{R}_+^{\ell_i} \setminus \{0\}, \quad \text{for all } \ y_i \in K_i, \end{cases}$$

where $\bar{x}_i$ is the ith component of $\bar{x}$.

$$\text{(SVVI)}_w \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } \ i \in I, \\ \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \notin \text{int } \mathbb{R}_+^{\ell_i}, \quad \text{for all } \ y_i \in K_i. \end{cases}$$

It is clear that every solution of (SVVI) is a solution of (SVVI)_w, but the converse is not true in general. (SVVI)_w is introduced and studied by Ansari, Schaible and Yao [10] in more general setting. They proved the existence of a solution of (SVVI)_w (for the infinite set I) by using a fixed point theorem for a family of multivalued maps. They used (SVVI)_w as a tool to prove the existence of a solution of system of vector optimization problems which includes the Nash equilibrium problem for vector valued functions.

Of course, if for each $i \in I$, $\ell_i = 1$, then (SVVI) and $(\text{SVVI})_w$ reduce to the (SVI) considered in Chapter 2.

One of the main motivations of this chapter is to study the existence of a solution of (SVVI) and $(\text{SVVI})_w$ so that by using the technique of Ansari, Schaible and Yao [10] one can derive the existence results for a solution of Nash equilibrium problem for vector valued functions. For this, we introduce the following *weighted variational inequality problem* over product of sets (for short, WVIP): find $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}\right)$ such that

$$\sum_{i \in I} W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } \; y_i \in K_i, \;\; i \in I,$$

where $\cdot$ denotes the inner product on $\mathbb{R}^{\ell_i}$ and $\mathbf{0}$ is the zero vector of $\mathbb{R}^{\ell_i}$. The solution set of (WVIP) is denoted by K^w.

We also introduce the following *system of weighted variational inequalities*:

$$(\text{SWVI}) \quad \begin{cases} \text{Find } \bar{x} \in K \;\text{ w. r. t. the weight vector }\; W = (W_1, \ldots, W_n) \\ \text{such that for each }\; i \in I, \;\; W_i \in \mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \;\text{ and} \\ W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all }\; y_i \in K_i. \end{cases}$$

If for each $i \in I$, $W_i \in \mathbb{T}_+^{\ell_i}$, then the solution of (WVIP) and (SWVI) are called *normalized*, where $\mathbb{T}_+^{\ell_i}$ is the simplex of $\mathbb{T}^{\ell_i}$, that is,

$$\mathbb{T}_+^{\ell_i} = \{u = (u_1, \ldots, u_{\ell_i}) \in \mathbb{R}_+^{\ell_i} : \sum_{j=1}^{\ell_i} u_j = 1\}.$$

We denote by K_s^w the solution set of (SWVI) and by K_n^w (respectively, K_{sn}^w) the normalized solution set of (WVIP) (respectively, (SWVI)).

The following lemma shows that the solution sets of (WVIP) and (SWVI) are equal.

Lemma 4.2.1. *For a given weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$), $K^w = K_s^w$ (respectively, $K^w = K_{sn}^w$).*

Proof. Obviously, $K_s^w \subseteq K^w$.

Let $\bar{x} \in K^w$, that is, $\bar{x} \in K$ be a solution of (WVIP), then

$$\sum_{i\in I} W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all } \ y_i \in K_i, \ \ i \in I.$$

For each $j \neq i$, let $y_j = \bar{x}_j$, then from the last inequality, we have that for each $i \in I$,

$$W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all } \ y_i \in K_i.$$

Hence $\bar{x} \in K_s^w$ and therefore $K^w = K_s^w$. □

Next we establish the following lemma which shows that (SVVI)$_w$ or (SVVI) can be solved by using (SWVI).

Lemma 4.2.2. *Each normalized solution $\bar{x} \in K$ with weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathbb{T}_+^{\ell_i}$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n}(\mathrm{int}\, \mathbb{T}_+^{\ell_i})$) of (SWVI) is a solution of (SVVI)$_w$ (respectively, (SVVI)).*

Proof. Let $\bar{x} \in K$ be a normalized solution of (SWVI) with weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathbb{T}_+^{\ell_i}$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n}(\mathrm{int}\ \mathbb{T}_+^{\ell_i})$). Suppose that $\bar{x} \in K$ is not a solution of (SVVI)$_w$ (respectively, (SVVI)). Then there would exist some $i \in I$ and a $y_i \in K_i$ satisfying

$$\langle F_i(\bar{x}), \bar{x}_i - y_i\rangle \in \mathrm{int}\ \mathbb{R}_+^{\ell_i} \ \ (\text{respectively}, \ \ \langle F_i(\bar{x}), \bar{x}_i - y_i\rangle \in \mathbb{R}_+^{\ell_i} \setminus \{0\}).$$

Since $W_i \in \mathbb{T}_+^{\ell_i}$ (respectively, $W_i \in \mathrm{int}\ \mathbb{T}_+^{\ell_i}$) for each $i \in I$, we have

$$W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i\rangle > 0, \quad \text{for all } \ i \in I,$$

which contradicts to our assumption that $\bar{x} \in K$ is a normalized solution of (SWVI). Hence $\bar{x} \in K$ is a solution of (SVVI)$_w$ (respectively, (SVVI)). □

In view of Lemmas 4.2.1 and 4.2.2, we have the following lemma.

Lemma 4.2.3. *Each normalized solution $\bar{x} \in K$ with weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathbb{T}_+^{\ell_i}$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n}(\mathrm{int}\, \mathbb{T}_+^{\ell_i})$) of (WVIP) is a solution of (SVVI)$_w$ (respectively, (SVVI)).*

4.3 Existence Results for (WVIP)

In this section, we first introduce different kinds of weighted monotonicities which are the extension of monotonicities defined in Chapter 2. Then we establish some existence results for a solution of (WVIP).

Definition 4.3.1. Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. A family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is said to be

(i) *weighted monotone w. r. t. the weight vector* W if for all $x, y \in K$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(y) - F_i(x), x_i - y_i \rangle \leq 0,$$

and *weighted strictly monotone w. r. t. the weight vector* W if the inequality is strict for all $x \neq y$;

(ii) *weighted pseudomonotone w. r. t. the weight vector* W if for all $x, y \in K$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i \rangle \leq 0 \quad \Rightarrow \quad \sum_{i\in I} W_i \cdot \langle F_i(y), x_i - y_i \rangle \leq 0,$$

and *weighted strictly pseudomonotone w. r. t. the weight vector* W if the second inequality is strict for all $x \neq y$;

(iii) *weighted maximal pseudomonotone w. r. t. the weight vector* W if it is weighted pesudomonotone and for all $x, y \in K$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(z), x_i - z_i \rangle \leq 0 \ \ \forall z \in \ ]x, y] \ \Rightarrow \ \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i \rangle \leq 0, \quad (4.3.1)$$

where z_i is the ith component of z, $]x, y] = \prod_{i\in I}]x_i, y_i]$ and $]x_i, y_i]$ denotes the line segment joining x_i and y_i but not containing x_i,

and *weighted maximal strictly pseudomonotone w. r. t. the weight vector* W if it is weighted strictly pseudomonotone and (4.3.1) holds.

Definition 4.3.2. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. A family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is said to be *weighted hemicontinuous w. r. t. the weight vector* W if for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \mapsto \sum_{i\in I} W_i \cdot \langle F_i(y + \lambda(x-y)), x_i - y_i\rangle$ is continuous.

Proposition 4.3.1. *If the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is weighted hemicontinuous and weighted pseudomonotone w. r. t. the weight vector $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$, then it is weighted maximal pseudomonotone w. r. t. the same weight vector $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$.*

Proof. Assume that for all $x, y \in K$,

$$\sum_{i\in I} W_i \cdot \langle F_i(z), x_i - z_i\rangle \le 0, \quad \text{for all } \ z \in \ ]x, y],$$

where z_i is the ith component of z. It follows that

$$\sum_{i\in I} W_i \cdot \langle F_i(y + \lambda(x-y)), x_i - (y_i + \lambda(x_i - y_i))\rangle \le 0, \quad \text{for all } \ \lambda \in (0,1]$$

which implies that

$$\sum_{i\in I} W_i \cdot \langle F_i(y + \lambda(x-y)), x_i - y_i\rangle \le 0, \quad \text{for all } \ \lambda \in (0,1].$$

By the weighted hemicontinuity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(y), x_i - y_i\rangle \le 0.$$

Hence the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is weighted maximal pseudomonotone. □

Now, we define another problem, which is closely related to the weighted variational inequality problem and can be termed as *Minty weighted variational inequality problem* (for short, MWVIP), of finding $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ such that

$$\sum_{i\in I} W_i \cdot \langle F_i(y), \bar{x}_i - y_i\rangle \le 0, \quad \text{for all } \ y_i \in K_i, \ \ i \in I.$$

The solution set of (MWVIP) is denoted by K_M^w.

The folowing lemma can be viewed as a generalization of Minty lemma (see, [80]).

Lemma 4.3.2. *If the family* $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ *of functions* $f_{i_j} : K \to X_i^*$ *is weighted maximal pseudomonotone w. r. t. the weight vector* $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$*, then* $K^w = K_M^w$.

Proof. It is obvious that $K^w \subseteq K_M^w$ by the weighted pseudomonotonicity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$.

Let $\bar{x} \in K_M^w$, then

$$\sum_{i\in I} W_i \cdot \langle F_i(y), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } \ y_i \in K_i, \quad i \in I.$$

Since for each $i \in I$, K_i is convex, we have $]\bar{x}_i, y_i] \subseteq K_i$, and therefore

$$\sum_{i\in I} W_i \cdot \langle F_i(z), \bar{x}_i - z_i \rangle \leq 0, \quad \text{for all } \ z_i \in \ ]\bar{x}_i, y_i], \quad i \in I.$$

By the weighted maximal pseudomonotonicity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } \ y_i \in K_i, \quad i \in I.$$

This shows that $\bar{x} \in K^w$ and hence $K^w = K_M^w$. □

Remark 4.3.1. In view of Proposition 4.3.1 and Lemma 4.3.2, we have that if the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is weighted hemicontinuous and weighted pseudomonotone w. r. t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$, then $K^w = K_M^w$.

Theorem 4.3.3. *For each* $i \in I$*, let* K_i *be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff)* X_i*. Let* $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ *be a weight vector and the family* $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ *of functions* $f_{i_j} : K \to X_i^*$ *be weighted maximal pseudomonotone w. r. t.* W*. Assume that there exist a nonempty, closed and compact subset* D *of* K *and* $\tilde{y} \in D$ *such that for all* $x \in K \setminus D$,

$\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$. Then there exists a solution $\bar{x} \in K$ of (WVIP) and hence it is a solution of (SWVI). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then there exists a normalized solution $\bar{x} \in K$ of (WVIP) and hence it is a solution of $(SVVI)_w$.

Proof. For each $x \in K$, define multivalued maps $S, T : K \to 2^K$ by

$$S(x) = \{y \in K : \sum_{i\in I} W_i \cdot \langle F_i(y), x_i - y_i\rangle > 0\}$$

and

$$T(x) = \{y \in K : \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle > 0\}.$$

Then it is clear that for each $x \in K, T(x)$ is convex. By weighted pseudomonotonicity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$, we have $S(x) \subseteq T(x)$ for all $x \in K$.

For each $y \in K$, the complement of $S^{-1}(y)$ in K is

$$[S^{-1}(y)]^c = \{x \in K : \sum_{i\in I} W_i \cdot \langle F_i(y), x_i - y_i\rangle \leq 0\}$$

is closed in K and hence $S^{-1}(y)$ is open in K. Therefore, $S^{-1}(y)$ is compactly open.

Assume that for all $x \in K$, $S(x)$ is nonempty. Then all the conditions of Theorem 1.2.5 are satisfied and therefore there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$. It follows that

$$0 = \sum_{i\in I} W_i \cdot \langle F_i(\tilde{x}), \tilde{x}_i - \tilde{x}_i\rangle > 0,$$

a contradiction. Hence there exists $\bar{x} \in K$ such that $S(\bar{x}) = \emptyset$. This implies that for all $y \in K$,

$$\sum_{i\in I} W_i \cdot \langle F_i(y), \bar{x}_i - y_i\rangle \leq 0,$$

that is, there exists $\bar{x} \in K$ w. r. t. the weight vector $W = W = (W_1, \ldots, W_n) \in \prod_{i=1}^n (\mathbb{R}_+^{\ell_i} \setminus \{0\})$ such that

$$\sum_{i\in I} W_i \cdot \langle F_i(y), \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all } \ y_i \in K_i \ \ i \in I.$$

By Lemma 4.3.2, $\bar{x} \in K$ is a solution of (WVIP) and so by Lemma 4.2.1 it is a solution of (SWVI).

If $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then $\bar{x} \in K$ is a normalized solution of (SWVI) and hence by Lemma 4.2.2 it is a solution of $(\mathrm{SVVI})_w$. Further, if $W \in \prod_{i\in I}(\mathrm{int}\ \mathbb{T}_+^{\ell_i})$ then again by Lemma 4.2.2 $\bar{x} \in K$ is a solution of $(\mathrm{SVVI})_w$. □

Remark 4.3.2. In Theorem 4.3.3, we have not assumed any kind of continuity assumption.

In view of Remark 4.3.1, we have the following result.

Corollary 4.3.4. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a vector and the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ be weighted hemicontinuous and weighted pseudomonotone w. r. t. W. Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$. Then there exists a solution $\bar{x} \in K$ of (WVIP) and hence it is a solution of (SWVI). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then there exists a normalized solution $\bar{x} \in K$ of (WVIP) and hence it is a solution of $(SVVI)_w$.*

Theorem 4.3.5. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector and the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ be weighted maximal strictly pseudomonotone w. r. t. W. Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$. Then there exists a unique solution of (WVIP) and hence it is a unique solution of (SWVI). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then there exists a unique normalized solution $\bar{x} \in K$ of (WVIP) which is also a uniuqe solution of $(SVVI)_w$.*

Proof. In view of Theorem 4.3.3, it is sufficient to show that (WVIP) has at most one solution. Suppose there exist two solutions x' and x'' of (WVIP), then we have

$$\sum_{i\in I} W_i \cdot \langle F_i(x''), x_i'' - x_i'\rangle \leq 0.$$

By the weighted strictly pseudomonotonicity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(x'), x_i'' - x_i' \rangle < 0, \quad \text{i.e.} \quad \sum_{i\in I} W_i \cdot \langle F_i(x'), x_i' - x_i'' \rangle > 0,$$

that is, x' is not a solution of (WVIP), a contradiction. □

In view of Corollary 4.3.4 and Theorem 4.3.5, we can easily derive the following result.

Corollary 4.3.6. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a vector and the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ be weighted hemicontinuous and weighted strictly pseudomonotone w. r. t. W. Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i \rangle > 0$. Then there exists a unique solution $\bar{x} \in K$ of (WVIP) and hence it is a unique solution of (SWVI). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then there exists a normalized unique solution $\bar{x} \in K$ of (WVIP) and hence it is a unique solution of $(SVVI)_w$.*

The following definition can be seen as an extension of Definition 2.3.7.

Definition 4.3.3. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. A family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is said to be *weighted B-pseudomonotone w. r. t. the weight vector* W if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha\in\Gamma}$ in K converging to x with

$$\limsup_\alpha \Big[\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - x_i \rangle \Big] \le 0$$

we have

$$\limsup_\alpha \Big[\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - y_i \rangle \Big] \ge \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i \rangle \quad \text{for all } \ y \in K.$$

Theorem 4.3.7. *For each $i \in I$, let K_i be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) X_i. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} (\mathbb{R}_+^{\ell_i} \setminus \{0\})$ be a weight vector and the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ be weighted B-pseudomonotone w. r. t. W such that for each $A \in \mathscr{F}(K)$, $x \mapsto \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle$ is lower semicontinuous on coA. Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K\setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$. Then there exists a solution $\bar{x} \in K$ of (WVIP) and hence it is a solution of (SWVI). Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then there exists a normalized solution $\bar{x} \in K$ of (WVIP) which also is a solution of $(SVVI)_w$.*

Proof. For each $x \in K$, let $T : K \to 2^K$ be the same as defined in the proof of Theorem 4.3.3. Then for all $x \in K$, $T(x)$ is convex. Let $A \in \mathscr{F}(K)$, then for all $y \in coA$,

$$[T^{-1}(y)]^c \cap coA = \{x \in coA : \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle \le 0\}$$

is closed in coA by lower semicontinuity of the map $x \mapsto \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle$ on coA. Hence $T^{-1}(y) \bigcap coA$ is open in coA.

Suppose that $x, y \in coA$ and $\{x^\alpha\}_{\alpha\in\Gamma}$ is a net in K converging to x such that

$$\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - (ty_i + (1-t)x_i)\rangle \le 0, \quad \text{for all } \ \alpha \in \Gamma \ \text{ and all } \ t \in [0,1].$$

For $t = 0$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - x_i\rangle \le 0, \quad \text{for all } \ \alpha \in \Gamma,$$

and therefore

$$\limsup_\alpha \big[\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - x_i\rangle\big] \le 0.$$

By the weighted B-pseudomonotonicity of the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$, we have

$$\limsup_\alpha \big[\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - y_i\rangle\big] \ge \sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle. \tag{4.3.2}$$

For $t = 1$, we have

$$\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - y_i\rangle \leq 0, \quad \text{for all} \quad \alpha \in \Gamma,$$

and therefore,

$$\limsup_{\alpha\in\Gamma}\big[\sum_{i\in I} W_i \cdot \langle F_i(x^\alpha), x_i^\alpha - y_i\rangle\big] \leq 0. \tag{4.3.3}$$

From (4.3.2) and (4.3.3), we get

$$\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - y_i\rangle \leq 0,$$

and thus $y \notin T(x)$.

Assume that for all $x \in K$, $T(x)$ is nonempty. Then all the conditions of Theorem 1.2.4 are satisfied. Hence there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, that is,

$$0 = \sum_{i\in I} W_i \cdot \langle F_i(\hat{x}), \hat{x}_i - \hat{x}_i\rangle > 0,$$

a contradiction. Thus there exists $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$, that is,

$$\sum_{i\in I} W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all} \quad y_i \in K_i \quad i \in I.$$

Hence $\bar{x}$ is a solution of (WVIP) and so by Lemma 4.2.1 it is a solution of (SWVI).

If $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then $\bar{x} \in K$ is a normalized solution of (SWVI) and hence by Lemma 4.2.2 it is a solution of $(\text{SVVI})_w$. Further, if $W \in \prod_{i\in I}(\text{int } \mathbb{T}_+^{\ell_i})$ then again by Lemma 4.2.2 $\bar{x} \in K$ is a solution of $(\text{SVVI})_w$. □

4.4 Weighted Variational Inequalities for Multivalued Maps

In this section, we introduce weighted variational inequalities for multivalued maps over product of sets, called *weighted generalized variational inequalities* and systems of generalized weighted variational inequalities. We also consider the systems of generalized vector variational inequalities. We show that the systems of generalized vector variational inequalities can be studied by using weighted generalized variational inequality problem over the product of sets or systems of weighted generalized variational inequalities. The monotonicities defined in Chapter 3 are extended and the existence results for a solution of weighted variational inequality problem over the product of sets, system of weighted generalized variational inequalities and systems of generalized vector variational inequalities are proved.

For each $i \in I$ and each $j = 1, \dots, \ell_i$, let $F_{i_j} : K \to 2^{X_i^*}$ be a multivalued map. For each $i \in I$, we denote

$$F_i(x) := F_{i_1}(x) \times \cdots \times F_{i_{\ell_i}}(x) \quad \text{for all} \;\; x \in K,$$

and

$$\langle u_i, x_i - y_i \rangle := \big(\langle u_{i_1}, x_i - y_i \rangle, \dots, \langle u_{i_{\ell_i}}, x_i - y_i \rangle \big),$$

where $\langle \cdot, \cdot \rangle$ denotes the continuous pairing between X_i^* and X_i. We further denote $F(x) := \{F_i(x)\}_{i \in I}$

Ansari, Schaible and Yao [11] introduced and studied the following problem of *system of generalized vector variational inequalities*: find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall \; y_i \in K_i, \;\; \exists \; \bar{u}_i \in F_i(\bar{x}) \;\; : \;\; \langle \bar{u}_i, \bar{x}_i - y_i \rangle \notin \text{int } \mathbb{R}_+^{\ell_i}. \tag{4.4.1}$$

They proved the existence of solution of above problem in more general setting (for the infinite index set I and for the moving cone) by using a maximal element theorem for a family of multivalued maps. They also studied the existence of a solution of Nash equilibrium problem for nondifferentiable (in some sense) vector valued functions and also for nondifferentiable and nonconvex vector valued functions.

We consider the following more general systems of generalized vector variational inequalities:

$$\text{(SGVVI)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \bar{u}_i \in F_i(\bar{x}) \text{ and} \\ \langle \bar{u}_i, \bar{x}_i - y_i \rangle \notin \mathbb{R}_+^{\ell_i} \setminus \{0\}, \quad \text{for all} \;\; y_i \in K_i, \end{cases}$$

where $\bar{x}_i$ is the ith component of $\bar{x}$ and 0 is the zero vector of $\mathbb{R}^{\ell_i}$.

$$\text{(SGVVI)}_w \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \bar{u}_i \in F_i(\bar{x}) \text{ and} \\ \langle \bar{u}_i, \bar{x}_i - y_i \rangle \notin \text{int } \mathbb{R}_+^{\ell_i}, \quad \text{for all} \;\; y_i \in K_i. \end{cases}$$

It is clear that every solution of (SGVVI) is a solution of $(\text{SGVVI})_w$, but the converse is not true in general.

Since for each $i \in I$, $u_i \in F_i(x)$ depends on $y_i \in K$ in the formulation (4.4.1), therefore (4.4.1) is called the *weak formulation of* $(\text{SGVVI})_w$.

To solve the (SGVVI) and $(\text{SGVVI})_w$, we introduce the following *weighted generalized variational inequality problem* over product of sets (for short, WGVIP): Find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ w. r. t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ such that

$$\sum_{i \in I} W_i \cdot \langle \bar{u}_i, \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } \ y_i \in K_i, \quad i \in I,$$

where $\cdot$ denotes the inner product on $\mathbb{R}^{\ell_i}$. The solution set of (WGVIP) is denoted by K^{wg}.

We also introduce the following *system of weighted generalized variational inequalities*:

$$(\text{SWGVI}) \quad \begin{cases} \text{Find } \bar{x} \in K \text{ and } \bar{u} \in F(\bar{x}) \\ \text{w. r. t. the weight vector } W = (W_1, \ldots, W_n) \\ \text{such that for each } i \in I, \ W_i \in \mathbb{R}_+^{\ell_i} \setminus \{0\} \text{ and} \\ W_i \cdot \langle \bar{u}_i, \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } \ y_i \in K_i. \end{cases}$$

If for each $i \in I$, $W_i \in \mathbb{T}_+^{\ell_i}$, then the solution of (WGVIP) or (SWGVI) is called *normalized.* We denote by K_s^{wg} the solution set of (SWGVI) and by K_{sn}^{wg} the normalized solution set of (SWGVI).

Definition 4.4.1. Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. A family $\{F_{i_j}\}_{i \in I, \ j=1}^{\ell_i}$ of multivalued maps $F_{i_j} : K \to 2^{X_i^*}$ is said to be

(i) *weighted generalized pseudomonotone w. r. t. the weight vector* W if for all $x, y \in K$ and for all $u \in F(x)$, $v \in F(y)$, we have

$$\sum_{i \in I} W_i \cdot \langle u_i, y_i - x_i \rangle \leq 0 \ \Rightarrow \ \sum_{i \in I} W_i \cdot \langle v_i, y_i - x_i \rangle \leq 0,$$

where u_i is the ith component of u;

(ii) *weighted generalized maximal pseudomonotone w. r. t. the weight vector W* if it is weighted generalized pesudomonotone and for all $x, y \in K$ and for all $u \in F(x)$, we have

$$\sum_{i \in I} W_i \cdot \langle s_i, z_i - x_i \rangle \leq 0 \ \ \forall s_i \in F_i(z), \ i \in I \text{ and } z \in \ ]x, y] \ \Rightarrow$$

$$\textstyle\sum_{i \in I} W_i \cdot \langle u_i, y_i - x_i \rangle \leq 0,$$

where z_i is the ith component of z;

(iii) *weighted u-hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \mapsto \sum_{i \in I} \langle F_i(x + \lambda z), z_i \rangle$ with $z = y - x$, is upper semicontinuous at 0, where z_i is the ith component of z.

Lemma 4.4.1. *Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. If the family $\{F_{i_j}\}_{i \in I, \ j=1}^{\ell_i}$ of multivalued maps $F_{i_j} : K \to 2^{X_i^*}$ is u-hemicontinuous and weighted generalized pseudomonotone, then it is weighted generalized maximal pseudomonotone.*

Proof. Suppose that for all $x, y \in K$ and for all $u \in F(x)$ we have

$$\sum_{i \in I} W_i \cdot \langle u_i, y_i - x_i \rangle > 0.$$

Set $z = ty + (1-t)x$ for $0 < t \leq 1$, that is $z \in]x, y]$. Then by weighted u-hemicontinuity of the family $\{F_{i_j}\}_{i \in I, \ j=1}^{\ell_i}$, there exists a $\delta > 0$ such that

$$\sum_{i \in I} W_i \cdot \langle s_i, y_i - x_i \rangle > 0, \quad \forall s \in F(z) \text{ and } t \in (0, \delta).$$

Since $t(y_i - x_i) = z_i - x_i$ for each $i \in I$, we have

$$\sum_{i \in I} W_i \cdot \langle s_i, z_i - x_i \rangle > 0, \quad \forall s_i \in F_i(z), \quad i \in I \text{ and } z \in \ ]x, y].$$

This completes the proof. □

Lemma 4.4.2. *Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. For each $\in I$ and each $j = 1, \ldots, \ell_i$, if f_{i_j} is a selection of F_{i_j} on K and $\bar{x} \in K$ is a solution of (WVIP) (respectively, SWVI) w. r. t. the weight vector W, then $(\bar{x}, \bar{u})$ is a solution of (WGVIP) (respectively, SWGVI) w. r. t. the same weight vector W with $\bar{u}_{i_j} \in F_{i_j}(\bar{x})$ for all $i \in I$ and all $j = 1, \ldots, \ell_i$.*

Proof. Let $\bar{x} \in K$ be a solution of (WVIP). Then

$$\sum_{i\in I} W_i \cdot \langle f_i(\bar{x}), \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all } \; y_i \in K_i, \; i \in I.$$

Let $\bar{u}_i = f_i(\bar{x})$ so that $\bar{u} = f(\bar{x})$. For each $i \in I$ and each $j = 1, \ldots, \ell_i$ since f_{i_j} is a selection of F_{i_j}, we have $\bar{u} \in F(\bar{x})$ and

$$\sum_{i\in I} W_i \cdot \langle \bar{u}_i, \bar{x}_i - y_i\rangle \leq 0, \quad \text{for all } \; y_i \in K_i, \; i \in I.$$

Hence $(\bar{x}, \bar{u})$ is a solution of (WGVIP). $\square$

Lemma 4.4.3. *Let $\bar{x} \in K$ be a normalized solution of (SWVI) with weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathbb{T}_+^{\ell_i}$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n}(int\ \mathbb{T}_+^{\ell_i})$) and for each $i \in I$ and each $J = 1, \ldots, \ell_i$, f_{i_j} is a selection of F_{i_j}, then $(\bar{x}, \bar{u})$ is a normalized solution of (SGVVI)$_w$ (respectively, (SGVVI)).*

Lemma 4.4.4. *Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $f_{i_j} : K \to X_i^*$ be a selection of a multivalued map $F_{i_j} : K \to 2^{X_i^*}$ on K. If the family $\{F_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of multivalued maps is weighted generalized maximal pseudomonotone w. r. t. the weight vector W, then the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions is weighted maximal pseudomonotone w. r. t. the same weight vector W.*

Theorem 4.4.5. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) X_i. Let $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. Assume that*

(i) the family $\{F_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of multivalued maps is weighted generalized maximal pseudomonotone w. r. t. the weight vector W;

(ii) for each $i \in I$ and each $j = 1, \ldots, \ell_i$, there exists a selection (not necessarily continuous) f_{i_j} of F_{i_j} on K;

(iii) there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$.

Then the solution set K^{wg} of (WGVIP) is nonempty and so is K_s^{wg}. Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then the normalized solution set of (SGVVI)$_w$ is nonempty.

Proof. From condition (ii) for each $i \in I$ and each $j = 1, \dots, \ell_i$, there is a function f_{i_j} such that $f_{i_j}(x) \in F_{i_j}(x)$ for all $x \in K$. By Lemma 4.4.4 we have $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ is weighted maximal pseudomonotone w. r. t. the weight vector W. Then all the conditions of Theorem 4.3.3 are satisfied and therefore there exists a solution $\bar{x} \in K$ of (WVIP). For each $i \in I$ and each $j = 1, \dots, \ell_i$, let $\bar{u}_{i_j} = f_{i_j}(\bar{x})$ so that $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Lemma 4.4.2 implies that $(\bar{x}, \bar{u})$ is a solution of (WGVIP), where $\bar{u} = (\bar{u}_i)_{i\in I}$. From lemma 4.4.3, $(\bar{x}, \bar{u})$ is a normalized solution of (SGVVIP)w. □

Theorem 4.4.6. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) X_i. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^{n} \left(\mathbb{R}_+^{\ell_i} \setminus \{0\}\right)$ be a weight vector. Assume that*

(i) the family of $\{F_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of multivalued maps is weighted generalized pseudomonotone w. r. t. the weight vector W;

(ii) for each $i \in I$ and each $j = 1, \dots, \ell_i$, there exists a continuous selection f_{i_j} of F_{i_j} on K;

(iii) there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$.

Then the solution set K^{wg} of (WGVIP) is nonempty and so is K_s^{wg}.

Proof. By condition (ii) for each $i \in I$ and each $j = 1, \dots, \ell_i$, there is a continuous function f_{i_j} such that $f_{i_j}(x) \in F_{i_j}(x)$ for all $x \in K$. Lemma 4.4.4 implies that the family $\{f_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of functions is weighted pseudomonotone w. r. t. the weight vector W and continuous, and so it is weighted pseudomonotone and weighted hemicontinuous w. r. t. the weight vector W. Then all the conditions of Corollary 4.3.4 are satisfied and therefore there exists a solution $\bar{x} \in K$ of (WVIP). For each $i \in I$ and each $j = 1, \dots, \ell_i$, let $\bar{u}_{i_j} = f_{i_j}(\bar{x}) \in F_{i_j}(\bar{x})$. Then by Lemma 4.4.2 we have $(\bar{x}, \bar{u})$ is a solution of (WGVIP), where $\bar{u} = (\bar{u}_i)_{i\in I}$. From lemma 4.4.3, $(\bar{x}, \bar{u})$ is a normalized solution of (SGVVIP)w. □

In view of Lemma 4.4.2 we have the following results.

Corollary 4.4.7. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) X_i. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^n (\mathbb{R}_+^{\ell_i} \setminus \{0\})$ be a weight vector. Assume that*

(i) the family $\{F_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of multivalued maps is weighted u-hemicontinuous and weighted generalized pseudomonotone w. r. t. the weight vector W;

(ii) for each $i \in I$ and each $j = 1, \dots, \ell_i$, there exists a selection (not necessarily continuous) f_{i_j} of F_{i_j} on K;

(iii) there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$.

Then the solution set K^{wg} of (WGVIP) is nonempty and so is K_s^{wg}. Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then the normalized solution set of $(SGVVI)_w$ is nonempty.

Corollary 4.4.8. *For each $i \in I$, let K_i be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) X_i. Let $W = (W_1, \dots, W_n) \in \prod_{i=1}^n (\mathbb{R}_+^{\ell_i} \setminus \{0\})$ be a weight vector. Assume that*

(i) the family $\{F_{i_j}\}_{i\in I,\ j=1}^{\ell_i}$ of multivalued maps is weighted generalized pseudomonotone w. r. t. the weight vector W;

(ii) for each $i \in I$ and each $j = 1, \dots, \ell_i$, there exists a continuous selection f_{i_j} of F_{i_j} on K;

(iii) there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i\rangle > 0$.

Then the solution set K^{wg} of (WGVIP) is nonempty and so is K_s^{wg}. Furthermore, if $W \in \prod_{i\in I} \mathbb{T}_+^{\ell_i}$ then the normalized solution set of $(SGVVI)_w$ is nonempty.

Chapter 5

Pareto Equilibria For Constrained Multiobjective Games

In this chapter, we study the existence of weighted Nash equilibria and Pareto equilibria for the constrained multiobjective games with or without involving Φ-condensing maps. Our results improve and unify the corresponding results for multiobjective games given in the literature.

5.1 Introduction

In 1950, Nash [105] (see, also [106]) introduced the concept of an equilibrium point in n-person games. Debreu [54] extended this concept of a Nash equilibrium point for n-person games to constrained equilibrium problems. In the last three decades, two problems, namely, n-person games and n-person games with constrained were extensively studied in the literature; See, for example, [23, 24] and references therein. In the recent past, much attention has been paid on the game theory with vector payoff; See, for example, [31, 35, 69, 127, 130, 131, 137, 140, 141] and references therein. The existence of Pareto equilibria is one of the fundamental problem in game theory.

Wang [131] introduced the concept of a weighted Nash equilibrium of a multiobjective game and proved that any normalized weighted Nash equilibrium is a weak Pareto equilibrium of a multiobjective game. He also formulated the weighted Nash equilibrium in terms of fixed points of a multivalued map. Wang [131], Yu and Yuan [137], and Yuan and Tarafdar [140] used this formulation to prove the existence of weighted Nash equilibrium and Pareto equilibrium of a multiobjective game by using some fixed point and minimax theorems. Recently, Ding [60] studied the multiobjective games with constrained correspondences. He proved the existence of weighted Nash equilibrium and Pareto equilibrium for the constrained multiobjective games in the setting of H-spaces.

The next section deals with the preliminaries. We also recall the formulations of multiobjective games and constrained multiobjective games. We define a weighted Nash equilibrium and a normalized weighted Nash equilibrium of a constrained multiobjective game. A relationship between a weighted Nash equilibrium of a constrained multiobjective game and an optimal solution of a constrained optimization problem is also given. In the third section of this chapter, we establish some existence results of a weighted Nash equilibrium and a Pareto equilibrium for the constrained multiobjective games with or without involving Φ-condensing maps. Our results improve and unify the corresponding results for multiobjective games given in the literature.

5.2 Preliminaries and Weighted Nash Equilibria

Through out this chapter, we follow the terminology of Ding [60] and, Yu and Yuan [137].

Let I be a finite index set, that is, $I = \{1, \ldots, n\}$ and for each $i \in I$, ℓ_i a positive integer. For each $i \in I$, let X_i be a nonempty subset of a topological vector space E_i, $X = \prod_{i \in I} X_i$ and $X^i = \prod_{j \in I, j \neq i} X_j$. For each $x \in X$, $x_i \in X_i$ denotes the ith coordinate, $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^i$ and we write $x = (x^i, x_i)$.

We consider a constrained game with finite players and multicriteria in its strategic form $\Gamma := (X_i, A_i, F_i)_{i \in I}$. For each player $i \in I$, X_i is its strategy set; $A_i : X^i \to 2^{X_i}$ is its constrained correspondence which restricts the strategies of the ith player to the subset $A_i(x^i) \subset X_i$ when all the players have chosen their strategies $x_j \in X_j$, $j \neq i$, and $F_i = (f_i^1, f_i^2, \ldots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is its payoff function (or say, loss function or multicriteria). In such a constrained multiobjective game, the other players influence player $j \in I$

(a) indirectly, by restricting js feasible strategies to $A_j(x^j)$,

(b) direct, by affecting js payoff function F_j.

If a choice $x = (x_1, \ldots, x_n) \in X$ is played, each player i is trying to mininize his/her payoff function $F_i(x) = (f_i^1(x), f_i^2(x), \ldots, f_i^{\ell_i}(x))$, which consists of noncommensurable outcomes. Each player i has a preference $\succcurlyeq_i$ over the outcome space $\mathbb{R}^{\ell_i}$. For each player $i \in I$, its preference $\succcurlyeq_i$ is given as follows:

$$z_1 \succcurlyeq_i z_2 \quad \text{if and only if} \quad z_1^j \geq z_2^j,$$

for each $j = 1, \ldots, \ell_i$, where $z_1 = (z_1^1, \ldots, z_1^{\ell_i})$ and $z_2 = (z_2^1, \ldots, z_2^{\ell_i})$ are any elements of $\mathbb{R}^{\ell_i}$. The players preference relations induce the preferences on X, defined for each player i and choose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in X$ by

$$x \succcurlyeq_i y, \quad \text{whenever} \quad F_i(x) \succcurlyeq_i F_i(y).$$

In the constrained multiobjective game, each player $i \in I$ is trying to minimize her/his own payoff according to her/his preferences.

If $A_i(x^i) = X_i$ for each $i \in I$ and for all $x^i \in X^i$, then the model of constrained multiobjective games reduced to the model of multicriteria games $G = (X_i, F_i)_{i \in I}$ considered and studied by Wang [130, 131], Ding [58], Yuan and Tarafdar [140] and Yu and Yuan [137] and references therein. If for each player $i \in I$, $F_i(x) = f_i(x)$, that is, $\ell_i = 1$, which consists of commensurable outcomes, then the model of the constrained multiobjective games reduced to the model of the constrained games (or say, metagames); See, for example, [23, 24] and references therein.

For the games with vector payoff functions (or multicriteria), as it is well known, in general, there does not exist a strategy $\bar{x} \in X$ to minimize (or equivalent to say, maximize) all f_i^js for each player $i \in I$; See, for example [139]. Hence, we need to recall some solution concepts for the constrained multicriteria games.

Definition 5.2.1. [60] A strategy $\bar{x}_i \in X_i$ of player i is called a *Pareto efficient strategy* (respectively, *a weak Pareto efficient strategy*) with respect to $\bar{x} \in X$ if $\bar{x}_i \in A_i(\bar{x}^i)$ and there is no strategy $y_i \in A_i(\bar{x}^i)$ such that

$$F_i(\bar{x}^i, \bar{x}_i) - F_i(\bar{x}^i, y_i) \in \mathbb{R}_+^{\ell_i} \setminus \{0\} \ \ \big(\text{respectively, } \ F_i(\bar{x}) - F_i(\bar{x}^i, y_i) \in \text{int } \mathbb{R}_+^{\ell_i}\big).$$

Definition 5.2.2. [60] A strategy combination $\bar{x} \in X$ is called a *Pareto equilibrium* (respectively, *a weak Pareto equilibrium*) of the constrained multiobjective game $\Gamma = (X_i, A_i, F_i)_{i \in I}$ if for each player i, $\bar{x}_i \in A_i(\bar{x}^i)$ is a Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to $\bar{x}$.

It is clear that each Pareto equilibrium is certainly a weak Pareto equilibrium, but the converse need not be true. We need the following concept which was introduced by Ding [60].

Definition 5.2.3. A strategy combination $\bar{x} \in X$ is called a *weighted Nash equilibrium* with weight combination $W = (W_1, W_2, \ldots, W_n)$ of a constrained multiobjective game $\Gamma = (X_i, A_i, F_i)_{i \in I}$ if for each player $i \in I$, we have that

(i) $\bar{x}_i \in A_i(\bar{x}^i)$;

(ii) $W_i \in \mathbb{R}_+^{\ell_i} \setminus \{0\}$; and

(iii) $W_i \cdot F_i(\bar{x}^i, \bar{x}_i) \leq W_i \cdot F_i(\bar{x}^i, y_i), \quad$ for all $y_i \in A_i(\bar{x}^i)$,

where $\cdot$ denotes the inner product in $\mathbb{R}^{\ell_i}$. In particular, when $W_i \in \mathbb{T}_+^{\ell_i}$, for all $i \in I$, the strategy $\bar{x} \in X$ is called a *normalized weighted Nash equilibrium* with respect to W.

From the above definition, it is easy to verify that a strategy $\bar{x} \in X$ is a weighted Nash equilibrium with respect to the weight vector $W = (W_1, W_2, \ldots, W_n)$ of the constrained multiobjective game $\Gamma = (X_i, A_i, F_i)$ if and only if $\bar{x} \in X$ is an optimal solution of the following constrained optimization problem:

$$(\text{COP}) \quad \begin{cases} \textbf{Find } \bar{x} \in X \textbf{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}^i) \textbf{ and} \\ W_i \cdot F_i(\bar{x}^i, \bar{x}_i) = \min_{y_i \in A_i(\bar{x}^i)} W_i \cdot F_i(\bar{x}^i, y_i). \end{cases}$$

We need the following lemma of Ding [60] which tells us that the existence problems of Pareto equilibrium for constrained multiobjective games can be reduced to the existence of the weighted Nash equilibrium under certain circumstances.

Lemma 5.2.1. *[60] Each normalized weighted Nash equilibrium $\bar{x} \in X$ with respect to a weight vector $W = (W_1, W_2, \ldots, W_n) \in \mathbb{T}_+^{k_1} \times \ldots \times \mathbb{T}_+^{k_n}$ (respectively, $W = (W_1, W_2, \ldots, W_n) \in int\ \mathbb{T}_+^{k_1} \times \ldots \times int\ \mathbb{T}_+^{k_n}$) for a constrained multiobjective game $\Gamma = (X_i, A_i, F_i)$ is a weak Pareto equilibrium (respectively, a Pareto equilibrium) of the game Γ.*

5.3 Existence of Weighted Nash Equilibria and Pareto Equilibria

In this section, we use some known fixed point theorems to prove the existence of a weighted Nash equilibrium and a Pareto equilibrium of a constrained multiobjective game.

Through out this section, unless otherwise specified, we assume that $X = \prod_{i\in I} X_i$ with the product topology.

Theorem 5.3.1. *Let $\Gamma = (X_i, A_i, F_i)$ be a constrained multiobjective game, where for each player $i \in I$, X_i is a nonempty, closed and convex subset of a Hausdorff topological vector space E_i, $A_i : X^i \to 2^{X_i}$ is the constrained correspondence, and $F_i = (f_i^1, f_i^2, \ldots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is the payoff function. Assume that there exists a weight vector $W = (W_1, W_2, \ldots, W_n)$ with $W_i \in \mathbb{R}^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) For each $i \in I$, $A_i : X^i \to 2^{X_i}$ is a multivalued map with nonempty and convex values and for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X. Further, we assume that the set $\mathscr{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A : X \to 2^X$ is Φ-condensing multivalued map defined as $A(x) = \prod_{i\in I} A_i(x^i)$ for all $x \in X$.

(ii) The function $(x, y) \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.

(iii) For each fixed $y \in X$, the mapping $x \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is upper semicontinuous on each compact subset of X.

(iv) For each fixed $x \in X$, the mapping $y \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is quasi-convex on X.

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W_i \in int\ \mathbb{T}_+^{\ell_i}$, then Γ has a Pareto equilibrium.

Proof. For the sake of simplicity, we define a bifunction $F : X \times X \to \mathbb{R}$ by

$$F(x,y) := \sum_{i \in I} W_i \cdot \left[F_i(x^i, x_i) - F_i(x^i, y_i)\right], \quad \text{for all} \;\; x, y \in X.$$

For each $x \in X$, define a multivalued map $P : X \to 2^X$ by

$$P(x) = \{y \in X : F(x,y) > 0\}.$$

Then by condition (iv), $P(x)$ is convex for all $x \in X$. From condition (ii) and (iii), for all $y \in X$, the complement of $P^{-1}(y)$ in X

$$[P^{-1}(y)]^C = \{x \in X : F(x,y) \le 0\}$$

is compactly closed in X and therefore $P^{-1}(y)$ is compactly open in X.

Since for each $i \in I$ and for all $x \in X$, $A_i(x^i)$ is nonempty and convex, we have $A(x) = \prod_{i \in I} A_i(x^i)$ is nonempty and convex. Also since for all $y \in X$, $A^{-1}(y) = \bigcap_{i \in I} A_i^{-1}(y_i)$ and $A_i^{-1}(y_i)$ is compactly open for each $i \in I$ and for all $y_i \in X_i$, we have $A^{-1}(y)$ is compactly open in X for all $y \in X$.

Now assume that for all $x \in \mathscr{D}$, $A(x) \cap P(x) \neq \emptyset$. Define another multivalued map $S : X \to 2^X$ by

$$S(x) := \begin{cases} A(x) \cap P(x), & \text{if } x \in \mathscr{D} \\ A(x), & \text{if } x \in K \setminus \mathscr{D}. \end{cases}$$

Then S has nonempty and convex values and by Lemma 3.2 in [60], $S^{-1}(y)$ is compactly open in X for all $y \in X$. Since $S(x) \subseteq A(x)$, for all $x \in K$ and A is Φ-condensing, by Remark 1.2.1 we have, S is also Φ-condensing. Hence by Theorem 1.2.5 along with Remark 1.2.2, there exists $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$. From the definition of $\mathscr{D}$ and S, we have $\{x \in X : x \in S(x)\} \subseteq \mathscr{D}$. Therefore, $\hat{x} \in \mathscr{D}$ and $\hat{x} \in A(\hat{x}) \cap P(\hat{x})$ and, in particular, we get

$$0 = F(\hat{x}, \hat{x}) = \sum_{i \in I} W_i \cdot \left[F_i(\hat{x}^i, \hat{x}_i) - F_i(\hat{x}^i, \hat{x}_i)\right] > 0,$$

a contradiction. Hence there exists $\bar{x} \in \mathscr{D}$ such that $A(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is, $\bar{x} \in A(\bar{x})$ and

$$F(\bar{x}, y) = \sum_{i \in I} W_i \cdot \left[F_i(\bar{x}^i, \bar{x}_i) - F_i(\bar{x}^i, y_i)\right] \le 0, \quad \text{for all} \;\; y \in A(\bar{x}).$$

For each $i \in I$ and for any given $y_i \in A_i(\bar{x}^i)$, let $y = (\bar{x}^i, y_i)$, then we have $y \in A(\bar{x})$ and it follows from last inequality that

$$W_i \cdot F_i(\bar{x}^i, \bar{x}_i) \leq W_i \cdot F_i(\bar{x}^i, y_i), \quad \text{for all } \; y_i \in A_i(\bar{x}^i).$$

This proves that for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}^i) \quad \text{and} \quad W_i \cdot F_i(\bar{x}^i, \bar{x}_i) = \min_{y_i \in A_i(\bar{x}^i)} W_i \cdot F_i(\bar{x}^i, y_i),$$

that is, $\bar{x} \in X$ is a weighted Nash equilibrium point for the constrained multiobjective game Γ with respect to weight vector W.

Lemma 5.2.1 shows that $\bar{x}$ is also a weak Pareto equilibrium of Γ, and Pareto equilibrium point of Γ if $W_i \in \operatorname{int} \mathbb{T}_+^{\ell_i}$ for all $i \in I$. □

Corollary 5.3.2. *Let $\Gamma = (X_i, A_i, F_i)$ be a constrained multiobjective game, where for each player $i \in I$, X_i is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space E_i, $A_i : X^i \to 2^{X_i}$ is the constrained correspondence, and $F_i = (f_i^1, f_i^2, \ldots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is the payoff function. Assume that there exists a weight vector $W = (W_1, W_2, \ldots, W_n)$ with $W_i \in \mathbb{R}^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) For each $i \in I$, $A_i : X^i \to 2^{X_i}$ is a multivalued map with nonempty and convex values and for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X. Further, we assume that the set $\mathscr{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A : X \to 2^X$ is a compact multivalued map defined as $A(x) = \prod_{i \in I} A_i(x^i)$ for all $x \in X$.

(ii) The function $(x, y) \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.

(iii) For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is upper semicontinuous on each compact subset of X.

(iv) For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is quasi-convex on X.

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W_i \in$ int $\mathbb{T}_+^{\ell_i}$, then Γ has a Pareto equilibrium.

Proof. Since for each $i \in I$, E_i is locally convex and $A : X \to 2^X$ is a compact multivalued map, by Remark 1.2.1 A is Φ-condensing map and the conclusion follows from Theorem 5.3.1. $\square$

For each $i \in I$ and for all $x^i \in X^i$, if $A_i(x^i) = X_i$ then we have the following existence results of weighted Nash equilibrium and Pareto equilibrium for multiobjective games.

Corollary 5.3.3. *Let $G = (X_i, F_i)$ be a multiobjective game, where for each player $i \in I$, X_i is a nonempty compact convex subset of a Hausdorff topological vector space E_i and $F_i = (f_i^1, f_i^2, \ldots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is the payoff function. Assume that there exists a weight vector $W = (W_1, W_2, \ldots, W_n)$ with $W_i \in \mathbb{R}^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) The function $(x, y) \mapsto \sum_{i\in} W_i \cdot F_i(x^i, y_i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.

(ii) For each fixed $y \in X$, the mapping $x \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is upper semicontinuous on each compact subset of X.

(iii) For each fixed $x \in X$, the mapping $y \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is quasi-convex on X.

Then the multiobjective game G has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W_i \in int\ \mathbb{T}_+^{\ell_i}$, then G has a Pareto equilibrium.

Proof. For each $i \in I$ and for all $x^i \in X^i$, let $A_i(x^i) = X_i$. Since for each $i \in I$, X_i is compact, the multivalued map $A : X \to 2^X$ defined as $A(x) = \prod_{i\in I} A_i(x^i)$ for all $x \in X$, is Φ-condensing. Then the conclusion follows from Theorem 5.3.1. $\square$

Corollary 5.3.4. *Let $G = (X_i, F_i)_{i\in I}$ be a given multiobjective game. For each $i \in I$, let X_i be a nonempty and convex subset of a Hausdorff topological vector space E_i. If there is a weight vector $W = (W_1, W_2, \ldots, W_n)$ with $W_i \in \mathbb{R}_+^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) The function $(x, y) \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.

(ii) For each fixed $y \in X$, the mapping $x \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is upper semicontinuous on each compact subset of X.

(iii) For each fixed $x \in X$, $\sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ is quasi-convex on X.

(iv) There exists a nonempty, compact and convex subset D_i of X_i such that for each $x \in X \setminus D$, there exists $\tilde{y}_i \in D_i$ such that $W_i \cdot [F_i(x^i, x_i)) - F_i(x^i, \tilde{y}_i)] > 0$, where $D = \prod_{i\in I} D_i \subseteq X$.

Then the multiobjective game G has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W_i \in int\ \mathbb{T}_+^{\ell_i}$, then the game G has a Pareto equilibrium.

Proof. For each $i \in I$, let $\{y_i^1, \dots, y_i^k\}$ be a finite subset of X_i. Let $C_i = co(D_i \cup \{y_i^1, \dots, y_i^k\})$. Then for each $i \in I$, C_i is nonempty, compact and convex. Then by Corollary 5.3.3, there exists $\bar{x} \in C = \prod_{i\in I} C_i$ such that for each $i \in I$,

$$W_i \cdot \left[F_i(\bar{x}^i, \bar{x}_i) - F_i(\bar{x}^i, y_i)\right] \leq 0, \quad \text{for all } \ y_i \in C_i.$$

From condition (iv), $\bar{x} \in D$. In particular, $\bar{x} \in D$ such that for each $i \in I$,

$$W_i \cdot \left[F_i(\bar{x}^i, \bar{x}_i) - F_i(\bar{x}^i, y_i^k)\right] \leq 0, \quad \text{for all } \ k.$$

For each $i \in I$ and for all $y_i \in X_i$, we now define

$$Q(y_i) = \{x \in D : W_i \cdot \left[F_i(x^i, x_i) - F_i(x^i, y_i)\right] \leq 0\}.$$

From conditions (i) and (ii), $Q(y_i)$ is closed for all $y_i \in X_i$. Hence every finite sub family of closed sets $Q(y_i)$ has nonempty intersection. Since D is compact, for each $i \in I$, $\bigcap_{y_i\in D_i} Q(y_i) \neq \emptyset$. And the result is proved. □

Remark 5.3.1. For each $i \in I$, if E_i is a reflexive Banach space equipped with the weak topology, the assumption (iv) in Corollary 5.3.4 can be replaced by the following condition.

(iv)′ There exists an $r > 0$ such that for all $x \in X$, $||x|| \geq r$, there exists $\tilde{y}_i \in X^i$, $||\tilde{y}_i||_i < r$ such that

$$W_i \cdot \left[F_i(x^i, x_i) - F_i(x^i, \tilde{y}_i)\right] > 0,$$

where $||\cdot||_i$ and $||\cdot||$ denote the norms on $\mathbb{R}^{\ell_i}$ and $\prod_{i=1}^n \mathbb{R}^{\ell_i}$, respectively.

Proof. Define $B_i^r = \{x_i \in X_i : ||x_i||_i \leq r\}$. Then B_i^r is a nonempty, compact and convex subset of X_i. By taking $D_i = B_i^r$ in Corollary 5.3.4, we get the conclusion. □

For each $i \in I$, when X_i is not necessarily Hausdorff and the multivalued map $A : X \to 2^X$ defined by $A(x) = \prod_{i \in I} A_i(x^i)$ for all $x \in X$ is not necessarily Φ-condensing, we have the following results.

Theorem 5.3.5. *Let $\Gamma = (X_i, A_i, F_i)$ be a constrained multiobjective game, where for each player $i \in I$, X_i is a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E_i, $A_i : X^i \to 2^{X_i}$ is the constrained correspondence, and $F_i = (f_i^1, f_i^2, \dots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is the payoff function. Assume that there exists a weight vector $W = (W_1, W_2, \dots, W_n)$ with $W_i \in \mathbb{R}^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) For each $i \in I$, $A_i : X^i \to 2^{X_i}$ is a multivalued map with nonempty and convex values and for each $y_i \in X_i$, $A_i^{-1}(y_i)$ is compactly open in X. Further, we assume that the set $\mathscr{D} = \{x \in X : x \in A(x)\}$ is compactly closed in X, where $A(x) = \prod_{i \in I} A_i(x^i)$ for all $x \in X$.

ii) The function $(x, y) \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is jointly lower semicontinuous on each compact subset of $X \times X$.

(iii) For each fixed $y \in X$, the mapping $x \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is upper semicontinuous on each compact subset of X.

(iv) For each fixed $x \in X$, the mapping $y \mapsto \sum_{i \in I} W_i \cdot F_i(x^i, y_i)$ is quasi-convex on X.

(v) For each $i \in I$, there exist a nonempty compact (not necessarily, convex) subset D_i of X_i and $\tilde{y}_i \in D_i$ such that for all $x \in X \setminus D$, $\tilde{y}_i \in A_i(x^i)$ and $W_i \cdot \big[F_i(x^i, x_i) - F_i(x^i, \tilde{y}_i)\big] > 0$, where $D = \prod_{i \in I} D_i \subseteq X$.

Then the constrained multiobjective game Γ has a weighted Nash equilibrium with the weight combination W and hence it has a weak Pareto equilibrium. Furthermore, if for all $i \in I$, $W_i \in int\ \mathbb{T}_+^{\ell_i}$, then Γ has a Pareto equilibrium.

Proof. It is easy to see that the condition (v) implies condition (iii) of Theorem 1.2.5, and the result follows from the proof of Theorem 5.3.1 by using Theorem 1.2.5. □

Corollary 5.3.6. *Let $G = (X_i, F_i)$ be a multiobjective game, where for each player $i \in I$, X_i is a nonempty convex subset of a topological vector space (not necessarily, Hausdorff) E_i and $F_i = (f_i^1, f_i^2, \dots, f_i^{\ell_i}) : X \to \mathbb{R}^{\ell_i}$ is the payoff function. Assume that there exists a weight vector $W = (W_1, W_2, \dots, W_n)$ with $W_i \in \mathbb{R}^{\ell_i} \setminus \{0\}$, for each $i \in I$, such that the following conditions are satisfied:*

(i) The function $(x, y) \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ *is jointly lower semicontinuous on each compact subset of* $X \times X$.

(ii) For each fixed $y \in X$, *the mapping* $x \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ *is upper semicontinuous on each compact subset of* X.

(iii) For each fixed $x \in X$, *the mapping* $y \mapsto \sum_{i\in I} W_i \cdot F_i(x^i, y_i)$ *is quasi-convex on* X.

(iv) For each $i \in I$, *there exist a nonempty compact (not necessarily, convex) subset* D_i *of* X_i *and* $\tilde{y}_i \in D_i$ *such that for all* $x \in X \setminus D$, $W_i \cdot \left[F_i(x^i, x_i) - F_i(x^i, \tilde{y}_i)\right] > 0$, *where* $D = \prod_{i\in I} D_i \subseteq X$.

Then the multiobjective game G *has a weighted Nash equilibrium with the weight combination* W *and hence it has a weak Pareto equilibrium. Furthermore, if for all* $i \in I$, $W_i \in int\ \mathbb{T}_+^{\ell_i}$, *then* G *has a Pareto equilibrium.*

Proof. For each $i \in I$ and for all $x^i \in X^i$, let $A_i(x^i) = X_i$, then we get the conclusion from Theorem 5.3.5. □

Remark 5.3.2. Corollaries 5.3.3, 5.3.4 and 5.3.6 generalize Theorems 3 and 4 in [137] in several ways.

Chapter 6

System of Generalized Vector Quasi-Equilibrium Problems

In this chapter, we introduce the concept of system of generalized vector quasi- equilibrium problems which includes system of generalized vector equilibrium problems, system of generalized implicit vector quasi-variational inequalities and system of vector quasi-equilibrium problems as special cases. By using known maximal element theorems for a family of multivalued maps, we prove the existence of a solution of system of generalized vector quasi-equilibrium problems. Several special cases are also discussed. As applications of our results, we derive the existence results for a solution of Debreu type equilibrium problem for vector-valued functions or constrained multiobjective games.

6.1 Introduction and Formulations

Let I be any index set and for each $i \in I$, let X_i be a Hausdorff topological vector space and K_i a nonempty convex subset of X_i. We set $K = \prod_{i \in I} K_i$, $X = \prod_{i \in I} X_i$ and $K^i = \prod_{j \in I,\, j \neq i} K_j$, and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we also write $x = (x^i, x_i)$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \to 2^{Y_i}$ be a multivalued map such that for each $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$, where int C denotes the interior of C. For each $i \in I$, let $F_i : K \times K_i \to 2^{Y_i}$ and

$A_i : K \to 2^{K_i}$ be multivalued maps with nonempty values. We consider the following *system of generalized vector quasi-equilibrium problems:*

$$\text{(SGVQEP)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ F_i(\bar{x}, y_i) \not\subseteq -\text{int } C_i(\bar{x}), \quad \text{for all } \ y_i \in A_i(\bar{x}). \end{cases}$$

If for each $i \in I$ and for all $x \in K$, $A_i(x) = K_i$, then (SGVQEP) reduces to the system of generalized vector equilibrium problems (for short, SGVEP) which is introduced and studied by Ansari et al [11] with applications to Nash equilibrium problem for vector-valued functions or multiobjective games.

If I is a singleton set, then (SGVQEP) reduces to a generalized vector quasi-equilibrium problem which contains generalized implicit vector quasi-variational inequality problem, generalized vector quasi-variational inequality and variational-like inequality problems and vector quasi-equilibrium problems as special cases. For further detail on generalized vector quasi-equilibrium problems and their applications, we refer [6] and references therein.

Examples of (SGVQEP):

For each $i \in I$, we denote by $L(X_i, Y_i)$ the space of all continuous linear operators from X_i into Y_i and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, let $T_i : K \to 2^{D_i}$ be a multivalued map with nonempty values.

(1) *System of Generalized Implicit Vector Quasi-Variational Inequalities*:
For each $i \in I$, let $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. The problem of system of generalized implicit vector quasi-variational inequalities (for short, SGIVQVIP) is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \ \ \exists \bar{u}_i \in T_i(\bar{x}) \ \ : \ \ \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}).$$

Setting for each $i \in I$,

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\}.$$

Then (SGVQEP) coincides with (SGIVQVIP).

For $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ for all $x \in K$ and for each $i \in I$, (SGIVQVIP) is called the *problem of system of generalized implicit quasi-variational inequalities.* Further, for all $x \in K$ and for each $i \in I$, $A_i(x) = K_i$, it is called the *problem of system of generalized implicit variational inequalities.* Ansari and Yao [20] studied such a problem with application to Nash equilibrium problem [105].

If I is a singleton set, (SGIVQVIP) reduces to the *generalized implicit vector quasi-variational inequality problem.*

The (SGIVQVIP) contains the following problems as special cases:

(i) For each $i \in I$, let $\theta_i : K \times D_i \to D_i$ and $\eta_i : K_i \times K_i \to X_i$ be bifunctions. If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \langle \theta_i(x, T_i(x)), \eta_i(y_i, x_i) \rangle =$$

$$\{\langle \theta_i(x, u_i), \eta_i(y_i, x_i) \rangle : u_i \in T_i(x)\},$$

then (SGIVQVIP) reduces to the *problem of system of generalized vector quasi-variational-like inequalities* (for short, SGVQVLIP) (I) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \theta_i(\bar{x}, \bar{u}_i), \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}),$$

where $\langle s_i, x_i \rangle$ denotes the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$. If I is a singleton set, then (SGVQVLIP)(I) becomes the *generalized vector quasi-variational-like inequality problem.* The strong solution (that is, $\bar{u}_i$ does not depend on y_i) of generalized vector quasi-variational-like inequality problem is studied by Chen et al [46] and Lee et al [94], see also references therein.

If for each $i \in I$, $\theta_i(x, u_i) = u_i$ for all $x \in K$, then (SGVQVLIP) (I) becomes the following problem denoted by (SGVQVLIP) (II): Find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i) \rangle \notin -\text{int } C_i(\bar{x}).$$

For $Y_i = \mathbb{R}$, $C_i(x) = \mathbb{R}_-$ and $A_i(x) = K_i$ for all $x \in K$ and for each $i \in I$, this problem is studied in [20] with application to the Nash equilibrium problem for nonconvex vector valued functions or multiobjective games for nonconvex functions.

(ii) If for each $i \in I$,

$$\psi_i(T_i(x), x_i, y_i) = \langle T_i(x), y_i - x_i \rangle = \{\langle u_i, y_i - x_i \rangle : u_i \in T_i(x)\},$$

then (SGIVQVIP) reduces to the *problem of system of generalized vector quasi-variational inequalities* (for short, SGVQVI P) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in T_i(\bar{x}) \quad : \quad \langle \bar{u}_i, y_i - \bar{x}_i \rangle \notin -\text{int } C_i(\bar{x}).$$

(2) *System of Vector Quasi-Equilibrium Problems* (for short, SVQEP): For each $i \in I$, let F_i be a single-valued map, then (SGVQEP) is equivalent to the following *system of vector quasi-equilibrium problems*:

$$\text{(SVQEP)} \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ F_i(\bar{x}, y_i) \notin -\text{int } C_i(\bar{x}), \quad \text{for all } \ y_i \in A_i(\bar{x}). \end{cases}$$

It is introduced and studied by Ansari et al [5] with applications to the Debreu type equilibrium problems for differentiable vector-valued functions or constrained multiobjective game.

The (SVQEP) contains the following problems as special cases:

(i) If for each $i \in I$, $F_i(x, y_i) = \langle T_i(x), y_i - x_i \rangle$, where T_i is a single-valued operator and x_i is the ith component of x, then (SVQEP) reduces to the *problem of system of vector quasi-variational inequalities* (for short, SVQVIP) which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\langle T_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\text{int } C_i(\bar{x}), \quad \text{for all } y_i \in A_i(\bar{x}).$$

If for each $i \in I$ and for all $x \in K$, $C_i(x) = \mathbb{R}_+$ and $Y_i = \mathbb{R}$, then (SVQVIP) reduces to the *system of quasi-variational inequalities* studied by Ansari et al [7].

(ii) For each $i \in I$, let $\varphi_i : K \to Y_i$ be a vector-valued function. If for each $i \in I$,

$$F_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x),$$

then (SVQEP) is equivalent to the following Debreu type equilibrium problem for vector-valued functions [54] or constrained multiobjective game:

$$\text{(DEP)} \begin{cases} \text{Find } \bar{x} \in K \text{ such that } \forall i \in I,\ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}), \text{ for all } y_i \in A_i(\bar{x}). \end{cases}$$

From the above examples/special cases, it is clear that (SGVQEP) is more general and unified format which contains many known problems as special cases. A comprehensive bibliography on vector equilibrium problems and vector variational inequalities can be found in a recent volume [71] edited by F. Giannessi.

In the next section, we establish some existence results for a solution to (SGVQEP) with or without involving Φ-condensing maps. As consequences, we prove the existence of a solution of many known problems mentioned above. Ansari et al [5] used (SVQEP) as a tool to prove the existence of a solution of Debreu type equilibrium problem for vector-valued but differentiable functions. As applications of our results, we derive the existence results for a solution of Debreu type equilibrium problem for vector-valued but nondifferentiable functions.

6.2 Existence Results

In this section, we establish some existence results for a solution of (SGVQEP) with or without involving Φ-condensing maps. As consequences, we drive the existence results for a solution of system of generalized implicit vector quasi-variational inequality problems, system of generalized vector quasi-variational-like inequality problems and system of generalized vector quasi-variational inequality problems.

Throughout this section, unless otherwise specified, we assume that I is any index set and for each $i \in I$, Y_i is a topological vector space, $K = \prod_{i \in I} K_i$, $C_i : K \to 2^{Y_i}$ is a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$. For each $i \in I$, we also assume that $A_i : K \to 2^{K_i}$ is a multivalued map such that for all $x \in K$, $A_i(x)$ is nonempty and convex, $A_i^{-1}(y_i)$ is open in K for all $y_i \in K_i$ and the set $\mathscr{F}_i := \{x \in K : x_i \in A_i(x)\}$ is closed in K, where x_i is the ith component of x.

Theorem 6.2.1. *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and let $F_i : K \times K_i \to 2^{Y_i}$ be a multivalued map with nonempty values. For each $i \in I$, assume that*

(i) for all $x \in K$, $F_i(x, x_i) \nsubseteq -\text{int } C_i(x)$, where x_i is the ith component of x;

(ii) for all $x \in K$, the set $\{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is convex;

(iii) for all $y_i \in K_i$, the set $\{x \in K : F_i(x, y_i) \nsubseteq -\text{int } C_i(x)\}$ is closed in K;

(iv) there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $F_i(x, \tilde{y}_i) \subseteq -\text{int } C_i(x)$.

Then (SGVQEP) has a solution.

Proof. For each $i \in I$, we define a multivalued map $Q_i : K \to 2^{K_i}$ by

$$Q_i(x) = \{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}, \quad \text{for all } x \in K.$$

By condition (ii), for each $i \in I$ and for all $x \in K$, $Q_i(x)$ is convex. Condition (iii) implies that for each $i \in I$ and for all $y_i \in K_i$, $Q_i^{-1}(y_i) = \{x \in K : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is open in K. Condition (i) implies that for each $i \in I$ and for all $x \in K$, $x_i \notin Q_i(x)$.

For each $i \in I$ and for all $x \in K$, we define another multivalued map $S_i : K \to 2^{K_i}$ by

$$S_i(x) := \begin{cases} A_i(x) \cap Q_i(x), & \text{if } x \in \mathscr{F}_i \\ A_i(x), & \text{if } x \in K \setminus \mathscr{F}_i. \end{cases}$$

Then, clearly for each $i \in I$ and for all $x \in K$, $S_i(x)$ is convex and $x_i \notin S_i(x)$. Since for each $i \in I$ and for all $y_i \in K_i$,

$$S_i^{-1}(y_i) = \left(A_i^{-1}(y_i) \cap Q_i^{-1}(y_i)\right) \bigcap \left((K \setminus \mathscr{F}_i) \cap A_i^{-1}(y_i)\right)$$

(see, for example, the proof of Lemma 2.3 in [59]) and $A_i^{-1}(y_i)$, $Q_i^{-1}(y_i)$ and $K \setminus \mathscr{F}_i$ are open in K, we have $S_i^{-1}(y_i)$ is open in K.

Condition (iv) of Theorem 1.2.6 is followed from condition (iv). Hence by Theorem 1.2.6, there exists $\bar{x} \in K$ such that $S_i(\bar{x}) = \emptyset$ for each $i \in I$. Since for each $i \in I$ and for all $x \in K$, $A_i(x)$ is nonempty, we have $A_i(\bar{x}) \cap Q_i(\bar{x}) = \emptyset$ for each $i \in I$. Therefore for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \text{ and } F_i(\bar{x}, y_i) \nsubseteq -\text{int } C_i(\bar{x}), \quad \text{for all } y_i \in A_i(\bar{x}).$$

This completes the proof. □

Next we establish an existence result for a solution to (SGVQEP) involving Φ-condensing maps.

Theorem 6.2.2. *For each $i \in I$, let K_i be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X_i, $F_i : K \times K_i \to 2^{Y_i}$ a multivalued map with nonempty values and let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i) - (iii) of Theorem 6.2.1 hold. Then (SGVQEP) has a solution.*

Proof. In view of Remark 1.2.3, it is sufficient to show that the multivalued map $S : K \to 2^K$ defined as $S(x) = \prod_{i \in I} S_i(x)$ for all $x \in K$, is ϕ-condensing, where S_i's are the same as defined in the proof of Theorem 6.2.1. By the definition of S_i, $S_i(x) \subseteq A_i(x)$ for each $i \in I$ and for all $x \in K$ and therefore $S(x) \subseteq A(x)$ for all $x \in K$. Since A is Φ-condensing, by Remark 1.2.1, we have S is also Φ-condensing. □

Definition 6.2.1. [19] Let W and Z be topological vector spaces and M a nonempty convex subset of W and let $P : M \to 2^Z$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A multivalued map $F : M \times M \to 2^Z \setminus \{\emptyset\}$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0, 1]$, we have either

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - P(x),$$

or

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_2) - P(x).$$

Remark 6.2.1. (a) If for each $i \in I$, F_i is C_x-quasiconvex-like, then the set $\{y_i \in K_i : F_i(x, y_i) \subseteq -\text{int } C_i(x)\}$ is convex, for each $x \in K$ (see, for example, the proof of Theorem 2.1 in [19]).
(b) If for each $i \in I$, X_i is locally convex Hausdorff topological vector space, the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\text{int } C_i(x)\}$ for all $x \in K$, is closed on K and for each $y \in K$, $F(., y)$ is upper semicontinuous on K, then condition (iii) of Theorem 6.2.1 is satisfied; See, for example, the proof of Theorem 2.1 in [19].

Definition 6.2.2. [8] Let W and Z be topological vector spaces, M a nonempty convex subset of W and D a nonempty subset of $L(W, Z)$. Let $T : M \to 2^D \setminus \{\emptyset\}$ and $P : M \to 2^Z$ be multivalued maps such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior. A function $\psi : D \times M \times M \to Z$ is called *$P(x)$-quasiconvex-like* if for all $x, y_1, y_2 \in M$ and $t \in [0, 1]$, we have either for all $u \in T(x)$,

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_1) - P(x),$$

or

$$\psi(u, x, ty_1 + (1-t)y_2) \in \psi(u, x, y_2) - P(x).$$

From Theorems 6.2.1 and 6.2.2, we derive the following existence result for a solution of (SGIVQVIP).

Corollary 6.2.3. *For each $i \in I$, let K_i be a nonempty convex subset of a locally convex topological vector space X_i and let D_i be a nonempty subset of $L(X_i, Y_i)$. For each $i \in I$, $T_i : K \to 2^{D_i}$ be an upper semicontinuous multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a vector-valued map. For each $i \in I$, assume that*

(i) the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-int\ C_i(x)\}$ for all $x \in K$, is closed on K;

(ii) for all $x \in K$ and $u_i \in T_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -int\ C_i(x)$, where x_i is the ith component of x;

(iii) ψ_i is $C_i(x)$-quasiconvex-like;

(iv) for all $y_i \in K_i$, the map $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ is upper semicontinuous on $D_i \times K_i$;

(v) there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\psi_i(u_i, x_i, \tilde{y}_i) \in -int\ C_i(x)$ for all $u_i \in T_i(x)$.

Then (SGIVQVIP) has a solution.

Proof. For each $i \in I$, we set

$$F_i(x, y_i) = \psi_i(T_i(x), x_i, y_i) = \{\psi_i(u_i, x_i, y_i) : u_i \in T_i(x)\}$$

for all $x \in K$ and $y_i \in K_i$. Then, all the conditions of Theorem 6.2.1 can easily be verified except for condition (iii). Hence we only need to prove that the set

$$\mathscr{D} = \{x \in K : \exists u_i \in T_i(x) \text{ s.t. } \psi_i(u_i, x_i, y_i) \notin -\text{int } C_i(x)\}$$

is closed in K for all $y_i \in K_i$. We prove it for a fixed i.

Let $\{x_\lambda\}$ be a net in $\mathscr{D}$ such that x_λ converges to $x^* \in K$. Then

$$\exists u_{i_\lambda} \in T_i(x_\lambda) \ \text{ s.t. } \ \psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \notin -\text{int } C_i(x_\lambda),$$

where x_{i_λ} is the ith component of x_λ, and therefore

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \in W_i(x_\lambda).$$

Let $\mathscr{K} = \{x_\lambda\} \cup \{x^*\}$. Then $\mathscr{K}$ is compact and $u_{i_\lambda} \in T_i(\mathscr{K})$ which is also compact. Therefore u_{i_λ} has a convergent subnet with limit u_{i_*}. Without loss of generality, we may assume that $\{u_{i_\lambda}\}$ converges to u_{i_*}. Then by upper semicontinuity of T, we have $u_{i_*} \in T_i(x^*)$. Since $\psi_i(.,.,y_i)$ is continuous and the graph of W_i is closed, we have

$$\psi_i(u_{i_\lambda}, x_{i_\lambda}, y_i) \ \text{ converges to } \ \psi_i(u_{i_*}, x_{i_*}, y_i) \in W_i(x^*),$$

and hence $\psi_i(u_{i_*}, x_{i_*}, y_i) \notin -\text{int } C_i(x^*)$. Therefore, $x^* \in \mathscr{D}$ and thus $\mathscr{D}$ is closed in K. This completes the proof. $\square$

Corollary 6.2.4. *For each $i \in I$, let $K_i, X_i, D_i, \psi_i, T_i$ and W_i be the same as in Corollary 6.2.3 and let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Assume that the conditions (i) - (iv) of Corollary 6.2.3 hold. Then (SGIVQVIP) has a solution.*

Let W and Z be Hausdorff topological vector spaces and σ be the family of all bounded subsets of W whose union is total in W, that is, the linear hull of $\bigcup\{U : U \in \sigma\}$ is dense in W. Let $\mathcal{B}$ be a neighborhood base of 0 in Z. When U runs through σ, V through $\mathcal{B}$, the family

$$M(U,V) = \{\xi \in L(W,Z) : \cup_{x\in U}\langle \xi, x\rangle \subseteq V\}$$

is a neighborhood base of 0 in $L(W,Z)$ for a unique translation-invariant topology, called the *topology of uniform convergence* on the sets $U \in \sigma$, or, briefly, the σ-topology (see [121], pp. 79-80).

In order to derive existence results for solutions of the (SGVQVLIP) and (SGVQVIP) from Corollaries 6.2.3 and 6.2.4, we need the following useful result due to Ding and Tarafdar [61].

Lemma 6.2.5. *Let W and Z be real Hausdorff topological vector spaces and $L(W,Z)$ be the topological vector space under the σ-topology. Then, the bilinear mapping $\langle .,. \rangle : L(W,Z) \times W \to Z$ is continuous on $L(W,Z) \times W$.*

Next we derive the existence results for a solution of (SGVQVLIP) by using Corollaries 6.2.3 and 6.2.4.

Corollary 6.2.6. *For each $i \in I$, let Y_i be a Hausdorff topological vector space and let K_i, X_i, D_i, T_i and W_i be the same as in Corollary 6.2.3. For each $i \in I$, let $L(X_i, Y_i)$ be equipped with the σ-topology. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be affine in the first argument and continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\langle u_i, \eta_i(\tilde{y}_i, x_i)\rangle \in -int\ C_i(x)$ for all $u_i \in T_i(x)$. Then (SGVQVLIP) has a solution.*

Corollary 6.2.7. *For each $i \in I$, let $K_i, X_i, Y_i, D_i, \eta_i, T_i$ and W_i be the same as in Corollary 6.2.6 and let $L(X_i, Y_i)$ be equipped with the σ-topology. For each $i \in I$, let $\eta_i : K_i \times K_i \to X_i$ be affine in the first argument and continuous in the second argument such that $\eta_i(x_i, x_i) = 0$ for all $x_i \in K_i$. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SGVQVIP) has a solution.*

In the last of this section, we have the following existence results for a solution of (SGVQVIP).

Corollary 6.2.8. *For each $i \in I$, let K_i, X_i, Y_i, D_i, T_i and W_i be the same as in Corollary 6.2.6 and let $L(X_i, Y_i)$ be equipped with the σ-topology. Assume that there exist a nonempty compact subset N of K and a nonempty compact convex subset B_i of K_i for each $i \in I$ such that for each $x \in K \setminus N$ there exist $i \in I$ and $\tilde{y}_i \in B_i$ satisfying $\tilde{y}_i \in A_i(x)$ and $\langle u_i, \tilde{y}_i - x_i \rangle \in -int\ C_i(x)$ for all $u_i \in T_i(x)$. Then (SGVQVIP) has a solution.*

Corollary 6.2.9. *For each $i \in I$, let K_i, X_i, Y_i, D_i, T_i and W_i be the same as in Corollary 6.2.6 and let $L(X_i, Y_i)$ be equipped with the σ-topology. Let the multivalued map $A = \prod_{i \in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SGVQVIP) has a solution.*

6.3 Applications

In this section we define the system of vector quasi-optimization problems which includes Debreu type equilibrium problem for vector-valued functions or constrained Nash equilibrium problem for vector-valued functions or constrained multiobjective game. By using the Clarke's generalized directional derivative, we prove the equivalence of solutions of system of generalized vector quasi-variational-like inequality problems and system of vector quasi-optimization problems. We use this equivalence and the results of previous section to derive the existence of solutions of systems of vector quasi-optimization problems for vector-valued functions, in particular, we establish the existence of a constrained multiobjective game.

Throughout this section, unless otherwise specified, we assume that the index set I is finite, that is, $I = \{1, \ldots, n\}$. For each $i \in I$, X_i and Y_i are finite dimensional Euclidean spaces $\mathbb{R}^{p_i}$ and $\mathbb{R}^{q_i}$, respectively, and K_i be a nonempty convex subset of X_i. For each $i \in I$, let $C_i : K \to 2^{Y_i}$ be a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with int $C_i(x) \neq \emptyset$ and $\mathbb{R}^{q_i}_+ \subseteq C_i(x)$. Let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ be defined as $A(x) = \prod_{i\in I} A_i(x)$, for all $x \in K$. For each $i \in I$, let $\varphi_i : K \to Y_i$ be a given vector-valued function. We consider the following *system of vector quasi-optimization problems* (in short, SVQOP) which is to find $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and for each $i \in I$,

$$\varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}) \quad \text{for all } y \in A(\bar{x}),$$

where $\varphi_i(x) = (\varphi_{i_1}(x), \varphi_{i_2}(x), \ldots, \varphi_{i_{q_i}}(x))$ and for each $l \in \mathscr{L} = \{1, \ldots, q_i\}$, $\varphi_{i_l} : K \to \mathbb{R}$ is a function.

We can choose $y \in A(x)$ in such a way that $y^i = \bar{x}^i$. Then (SVQOP) reduces to the *Debreu type equilibrium problem for vector-valued functions* or constrained multiobjective game which is to find $\bar{x} \in K$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}) \quad \text{for all } y_i \in A_i(\bar{x}).$$

It is clear that every solution of (SVQOP) is also a solution of the Debreu type equilibrium problem for vector-valued functions, but the converse need not be true.

Now we recall some definitions.

Definition 6.3.1. A real-valued function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *locally Lipschitz* if for any $z \in \mathbb{R}^p$ there exist a neighborhood $N(z)$ of z and a positive constant k such that

$$|f(x) - f(y)| \leq k||x - y|| \quad \text{for all } x, y \in N(z).$$

The Clarke *generalized directional derivative* [50] of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

The Clarke *generalized gradient* [50] of a locally Lipschitz function f at x is defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^p : f^0(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^p\}.$$

If f is convex, then the Clarke generalized gradient coincides with the subdifferential of f in the sense of convex analysis [119].

The generalized invex function was introduced by Craven [52] as a generalization of invex functions [76].

Definition 6.3.2. A locally Lipschitz function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *generalized invex at x w.r.t. a given function $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$* if

$$f(y) - f(x) \geq \langle \xi, \eta(y, x) \rangle \quad \text{for all } \xi \in \partial f(x) \text{ and } y \in \mathbb{R}^p.$$

For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be a locally Lipschitz function and let $x \in K$, $x_j \in K_j$. Following Clarke [50], the *generalized directional derivative at x_j in the direction $d_j \in K_j$* of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ denoted by $\phi_{ij}^0(x; d_j)$ is

$$\phi_{ij}^0(x; d_j) = \lim_{\substack{y_j \to x_j \\ t \downarrow 0}} \sup \frac{1}{t}\{\phi_i(x_1, \ldots, x_{j-1}, y_j + td_j, x_{j+1}, \ldots, x_n)$$

$$-\phi_i(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)\}.$$

The *partial generalized gradient* [50] of the function $\phi_i(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$ at x_j is defined as follows:

$$\partial_j \phi_i(x) = \{\xi_j \in X_j : \phi_{ij}^0(x; d_j) \geq \langle \xi_j, d_j \rangle \text{ for all } d_j \in K_j\}.$$

Lemma 6.3.1. *[50] For each $i \in I$, let $\phi_i : K \to \mathbb{R}$ be locally Lipschitz. Then for each $i \in I$, the multivalued map $\partial_i \phi_i$ is upper semicontinuous.*

Definition 6.3.3. For each $i \in I$, $\phi_i : K \to \mathbb{R}$ is called *generalized invex* at x w.r.t. a given function $\eta_i : K_i \times K_i \to \mathbb{R}^{p_i}$ if

$$\phi_i(y) - \phi_i(x) \geq \langle \xi_i, \eta_i(y_i, x_i) \rangle \quad \text{for all } \xi_i \in \partial_i \phi_i(x) \text{ and } y \in K.$$

Proposition 6.3.2. *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}_+^{\ell_i}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$. Then any solution of (SGVQVLIP) is a solution of (SVQOP) with $T_i(x) = \partial_i\varphi_i(x)$ for each $i \in I$ and for all $x \in K$ where $\partial_i\varphi_i(x) = (\partial_i\varphi_{i_1}(x), \partial_i\varphi_{i_2}(x), \ldots, \partial_i\varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$.*

Proof. For the sake of simplicity, we denote by $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$ where $u_{i_l} \in \partial_i\varphi_{i_l}(x)$ for all $l \in \mathscr{L}$, and

$$\langle u_i, \eta_i(y_i, x_i)\rangle = (\langle u_{i_1}, \eta_{i_1}(y_i, x_i)\rangle, \ldots, \langle u_{i_{q_i}}, \eta_{i_{q_i}}(y_i, x_i)\rangle) \in \mathbb{R}^{q_i}.$$

Assume that $\bar{x} \in K$ is a solution of (SGVVLIP). Then for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_{i_l} \in \partial_i\varphi_{i_l}(\bar{x}) \text{ for all } l \in \mathscr{L} \text{ such that}$$

$$(\langle \bar{u}_{i_1}, \eta_{i_1}(y_i, \bar{x}_i)\rangle, \ldots, \langle \bar{u}_{i_{q_i}}, \eta_{i_{q_i}}(y_i, \bar{x}_i)\rangle) \notin -\text{int } C_i(\bar{x}).$$

We can rewrite this as

$$\forall y_i \in A_i(\bar{x}), \quad \exists \bar{u}_i \in \partial_i\varphi_i(\bar{x}) \quad : \quad \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle \notin -\text{int } C_i(\bar{x}). \tag{6.3.1}$$

Since for each $i \in I$ and for all $l \in \mathscr{L}$, φ_{i_l} is generalized invex w.r.t. η_{i_l}, we have

$$\varphi_{i_l}(y) - \varphi_{i_l}(\bar{x}) \geq \langle u_{i_l}, \eta_{i_l}(y_i, \bar{x}_i)\rangle \quad \text{for all } u_{i_l} \in \partial_i\varphi_{i_l}(\bar{x}) \text{ and } y \in A(\bar{x}),$$

that is, for each $i \in I$

$$\varphi_i(y) - \varphi_i(\bar{x}) \geq \langle u_i, \eta_i(y_i, \bar{x}_i)\rangle \quad \text{for all } u_i \in \partial_i\varphi_i(\bar{x}) \text{ and } y \in A(\bar{x}).$$

Therefore for each $i \in I$ and for all $u_i \in \partial_i\varphi_i(\bar{x})$, we have

$$\begin{aligned} \varphi_i(y) - \varphi_i(\bar{x}) &\in \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle + \mathbb{R}_+^{q_i} \\ &\subseteq \langle \bar{u}_i, \eta_i(y_i, \bar{x}_i)\rangle + \text{int } C_i(\bar{x}). \end{aligned} \tag{6.3.2}$$

From (6.3.1) and (6.3.2), we have $\varphi_i(y) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x})$. Hence $\bar{x} \in K$ is a solution of (SVQOP). $\square$

Rest of the section, unless otherwise specified, $\partial_i\varphi_i(x)$ and $\langle u_i, \eta_i(y_i, x_i)\rangle$ are the same as defined in Proposition 6.3.2

Theorem 6.3.3. *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Assume that there exists $r > 0$ such that for all $x \in K$, $||x|| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $||\tilde{y}_i||_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and*

$$\langle u_i, \eta_i(\tilde{y}_i, x_i)\rangle \in -int\ C_i(x) \quad \text{for all } u_i \in \partial_i\varphi_i(x),$$

where $||\cdot||$ and $||\cdot||_i$ denote the norms on X and X_i, respectively. Then (SVQOP) has a solution.

Proof. For each $i \in I$ and for all $x \in K$, $T_i(x) = \partial_i\varphi_i(x)$ is an upper semicontinuous multivalued map by Lemma 6.3.1. It is easy to check that all the conditions of Corollary 6.2.6 are satisfied. Hence from Corollary 6.2.6 and Proposition 6.3.2 it follows that (SVQOP) has a solution. □

Theorem 6.3.4. *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be generalized invex w.r.t. $\eta_{i_l} : K_i \times K_i \to X_i$ such that η_{i_l} is affine in the first argument, continuous in the second argument and $\eta_{i_l}(x_i, x_i) = 0$ for all $x_i \in K_i$. Let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SVQOP) has a solution.*

In the next two corollaries, we set $\varphi_i(x) = (\varphi_{i_1}(x), \ldots, \varphi_{i_{q_i}}(x)) \in \mathbb{R}^{q_i}$, $u_i = (u_{i_1}, \ldots, u_{i_{q_i}})$, $\langle u_i, y_i - x_i\rangle = (\langle u_{i_1}, y_i - x_i\rangle, \ldots, \langle u_{i_{q_i}}, y_i - x_i\rangle) \in \mathbb{R}^{q_i}$ and $\partial_i\varphi_i(x) = (\partial_i\varphi_{i_1}(x), \partial_i\varphi_{i_2}(x), \ldots, \partial_i\varphi_{i_{q_i}}(x)) \in \mathbb{R}^{p_i \times q_i}$, where $\partial_i\varphi_{i_j}(x)$ $(j = 1, \ldots, q_i)$ is the partial subdifferential in the sense of convex analysis.

Corollary 6.3.5. *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be convex and lower semicontinuous Assume that there exists $r > 0$ such that for all $x \in K$, $||x|| > r$, there exist $i \in I$ and $\tilde{y}_i \in K_i$ with $||\tilde{y}_i||_i \leq r$ satisfying $\tilde{y}_i \in A_i(x)$ and*

$$\langle u_i, \tilde{y}_i - x_i\rangle \in -int\ C_i(x) \quad \text{for all } u_i \in \partial_i\varphi_i(x),$$

where $||\cdot||$ and $||\cdot||_i$ denote the norms on X and X_i, respectively. Then (SVQOP) has a solution.

Corollary 6.3.6. *For each $i \in I$ and for all $l \in \mathscr{L}$, let $\varphi_{i_l} : K \to \mathbb{R}$ be convex and lower semicontinuous on K. Let the multivalued map $A = \prod_{i\in I} A_i : K \to 2^K$ defined as $A(x) = \prod_{i\in I} A_i(x)$ for all $x \in K$, be Φ-condensing. Then (SVQOP) has a solution.*

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